Sections 1A & 1C

MTH 204
Fall 2008
Exam 1

Name: Key

Section: A or C (circle one)

Read the directions carefully.
Each question is worth 20 points,
with a maximum of 100 points possible.
Write <u>neatly</u> in pencil and **<u>show all your work</u>**
<u>(you will only get credit for what you put on paper)</u>.
Please do not share calculators during the test.
If you have trouble during the test, feel free to ask me for help.

Score:________

1. Consider the differential equation $\dfrac{dy}{dx} = (y-2)^2(y^2-9)$.

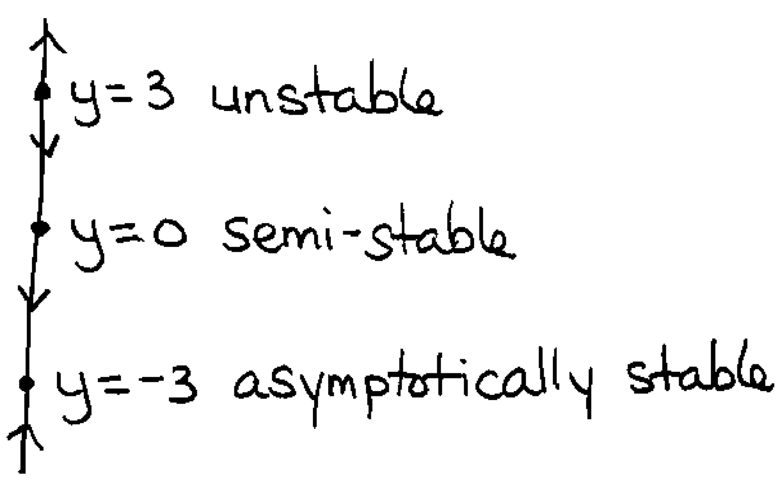
   A. Find the order of the differential equation. Is the differential equation (non)linear, (non)autonomous, and (non)separable? Explain why.

   1st order
   nonlinear - coefficient depends on y.
   autonomous - x doesn't show up in the equation
   Separable - can be written as $\frac{dy}{dx} = f(x,y) = g(x)h(y)$ where $g(x)=1$.

   B. Find the critical points for the differential equation. Classify each point as either asymptotically stable, unstable, or semi-stable. Draw the appropriate phase portrait.

   CP: $y = -3, 2, 3$

Int	TV	+/−	↑/↓
$(-\infty,-3)$	-4	+	↑
$(-3,2)$	-1	−	↓
$(2,3)$	2.5	−	↓
$(3,\infty)$	4	+	↑



2. Solve the initial value problem $\frac{dy}{dx} = x(y - y^2)$; $y(0) = \frac{1}{2}$ (an implicit solution is acceptable)

$\frac{dy}{dx} = xy(1-y)$ 1st order, nonlinear, autonomous, separable

$$\Rightarrow \int \frac{dy}{y(1-y)} = \int x\,dx$$

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y} = \frac{1}{y} + \frac{1}{1-y}$$

$\Rightarrow 1 = A(1-y) + By$

$y=0 \Rightarrow 1 = A$

$y=1 \Rightarrow 1 = B$

$$\Rightarrow \int \left(\frac{1}{y} + \frac{1}{1-y}\right) dy = \int x\,dx \qquad \begin{matrix} u = 1-y \\ du = -dy \end{matrix}$$

$\ln|y| - \ln|1-y| = \frac{1}{2}x^2 + C$

$\ln\left|\frac{y}{1-y}\right| = \frac{1}{2}x^2 + C$

$y(0) = \frac{1}{2} \Rightarrow \ln\left|\frac{1/2}{1-1/2}\right| = 0 + C \Rightarrow C = 0$

$\ln\left|\frac{y}{1-y}\right| = \frac{1}{2}x^2$ implicit solution

or $\frac{y}{1-y} = e^{\frac{1}{2}x^2} \Rightarrow y = (1-y)e^{\frac{1}{2}x^2} = e^{\frac{1}{2}x^2} - e^{\frac{1}{2}x^2}y$

$\Rightarrow \left(1 + e^{\frac{1}{2}x^2}\right)y = e^{\frac{1}{2}x^2}$

$\Rightarrow y(x) = \frac{e^{\frac{1}{2}x^2}}{1 + e^{\frac{1}{2}x^2}}$ explicit solution

3. Consider the differential equation $xy' + (x+1)y = e^{-x}\sin(2x)$.

A. Find the order of the differential equation. Is the differential equation (non)linear, (non)autonomous, and (non)separable? Explain why.

1st order

linear - equation of the form $a_1(x)y' + a_0(x)y = g(x)$

nonautonomous - x is in the equation

nonseparable - cannot be rewritten as

$$dy/dx = g(x)h(y)$$

B. Give a maximal interval I over which the solution is defined.

Std form: $y' + \left(\frac{x+1}{x}\right)y = \frac{e^{-x}\sin(2x)}{x}$ $\left\{ \begin{array}{l} x > 0 \text{ or} \\ x < 0 \end{array} \right.$

C. Find an explicit solution to the differential equation.

$$IF = e^{\int P(x)dx} = e^{\int (1+\frac{1}{x})dx} = e^{x+\ln x} = xe^x, \; x > 0$$

$$xe^x\left[y' + \left(\frac{x+1}{x}\right)y\right] = xe^x\left(\frac{e^{-x}\sin(2x)}{x}\right) = \sin(2x)$$

$$xe^x y' + (x+1)e^x y = \sin(2x)$$

$$\frac{d}{dx}[xe^x y] = \sin(2x)$$

$$\int \frac{d}{dx}[xe^x y]dx = \int \sin(2x)dx$$

$$xe^x y = -\frac{1}{2}\cos(2x) + C$$

$$y(x) = \frac{-\cos(x)}{2xe^x} + \frac{C}{xe^x}$$

4. Newton's law of cooling states that the rate of change in the temperature of a body is *proportional* to the difference in the temperature of that body and temperature of the surrounding medium. Now suppose that a murder victim is discovered at midnight with a recorded temperature of 31° C. An hour later, the temperature of the victim is 29° C. Assume that the temperature of the surrounding air remains a constant 21° C. Calculate the victim's time of death. Note: the "normal" temperature of a living person is 37° C.

$$\frac{dT}{dt} = K(T-21) \qquad \begin{cases} T(0)=31 \\ T(1)=29 \end{cases}$$

1st order, linear, autonomous, separable, nonhomogeneous

Methods $\begin{cases} \text{SOV} \checkmark \\ \text{IF} \end{cases}$

$$\Rightarrow \int \frac{dT}{T-21} = K\int dt$$

$$\ln|T-21| = Kt + c$$

$$T(t) - 21 = e^{Kt+c}$$

$$= c_1 e^{Kt}$$

$$T(t) = 21 + c_1 e^{Kt}$$

$$T(0) = 31 = 21 + c_1 e^{0}$$

$$\Rightarrow c_1 = 10$$

$$\Rightarrow T(t) = 21 + 10e^{Kt}$$

$$T(1) = 29 = 21 + 10e^{K}$$

$$\Rightarrow e^{K} = \frac{4}{5}$$

$$\Rightarrow K = \ln\left(\frac{4}{5}\right)$$

$$\Rightarrow T(t) = 21 + 10e^{\ln(4/5)t}$$

Now let $T(t_1) = 37 = 21 + 10e^{\ln(4/5)t_1}$

$$\Rightarrow \ln\left(\frac{4}{5}\right)t_1 = \ln\left(\frac{8}{5}\right)$$

$$\Rightarrow t_1 = \frac{\ln(8/5)}{\ln(4/5)} \approx -2.106 \approx -2\text{hrs } 6\text{mins}$$

$\Rightarrow$ Time of death: 9:54 PM.

5. Use reduction of order to find a second linearly independent solution to the differential equation $xy''+(1-2x)y'+(x-1)y=0, \quad x>0$, where $y_1(x)=e^x$ is a known solution.

(**<u>NO POINTS</u>** for the integral formula).

1. Assume $y_2(x) = u(x)\,y_1(x) = u(x)e^x$ is a solution

$$y_2' = u'e^x + ue^x$$
$$y_2'' = u''e^x + 2u'e^x + ue^x$$

2. Plug in y_2

$$e^x[x(u''+2u'+u)+(1-2x)(u'+u)+(x-1)u]=0$$

3. Regroup by u

$$xu''+(2x+1-2x)u'+(x+1-2x+x-1)u=0$$
$$xu''+u''=0$$

4. Change of Variables

Let $\begin{cases} w=u' \\ w'=u'' \end{cases} \Rightarrow x\dfrac{dw}{dx}+w=0$

1st order, linear, Hom, VC, Sep nonauto

Methods $\begin{cases} \text{SoV} \checkmark \\ \text{IF} \end{cases}$

5. Solve for w

$$\int\frac{dw}{w} = -\int\frac{dx}{x} \Rightarrow e^{\ln|w|} = e^{-\ln|x|+K}$$
$$\Rightarrow w = K_1x^{-1} = u'$$

6. Solve for u

$$u = K_1\int\frac{dx}{x} = K_1\ln x + K_2 \qquad \text{Pick } \begin{cases} K_1=1 \\ K_2=0 \end{cases}$$
$$= \ln x$$

7. Find $y_2(x) = u(x)\,y_1(x)$

$$= e^x \ln x$$

Bonus (10 points): Consider the differential equation $\frac{dy}{dx} = \sqrt{y - x^2 + 1}$. Determine a region R in the xy-plane for which the differential equation would have a unique solution through each point (x_0, y_0). Sketch the region.

$$\frac{dy}{dx} = f(x,y) = \sqrt{y-x^2+1}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y-x^2+1}}$$

$$R = \{(x,y) : y - x^2 + 1 > 0\}$$
$$= \{(x,y) : y > x^2 - 1\}$$