MTH 204
Fall 2006
Exam 1

Name: Key

Section: B

Read the directions carefully.
Each question is worth 20 points.
Write neatly in pencil and **show all your work**
(you will only get credit for what you put on paper).
Please do not share calculators during the test.
If you have trouble during the test, feel free to ask me for help.

1. Consider the differential equation $\frac{dy}{dx} = \sqrt{xy} = f(x,y)$.

A. Determine a region R in the xy-plane for the differential equation would have a unique solution through each point (x_0,y_0).

1. $f(x,y) = \sqrt{xy}$

   $x \geq 0,\ y \geq 0$

   $x \leq 0,\ y \leq 0$

2. $\frac{\partial f(x,y)}{\partial y} = \frac{1}{2}\sqrt{\frac{x}{y}}$

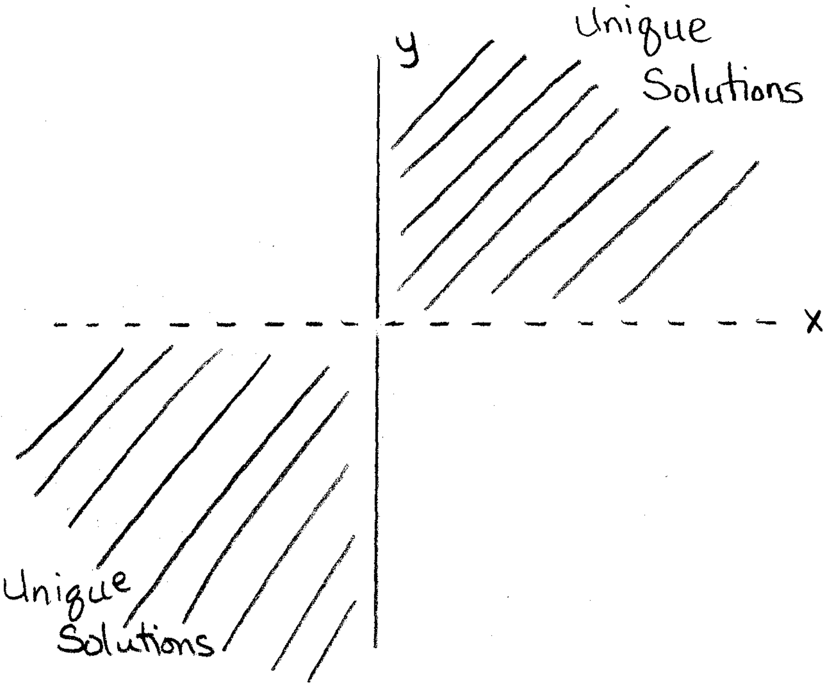
   $x \geq 0,\ y > 0$

   $x \leq 0,\ y < 0$

3. Region R in the xy-plane

   $x \geq 0,\ y > 0$

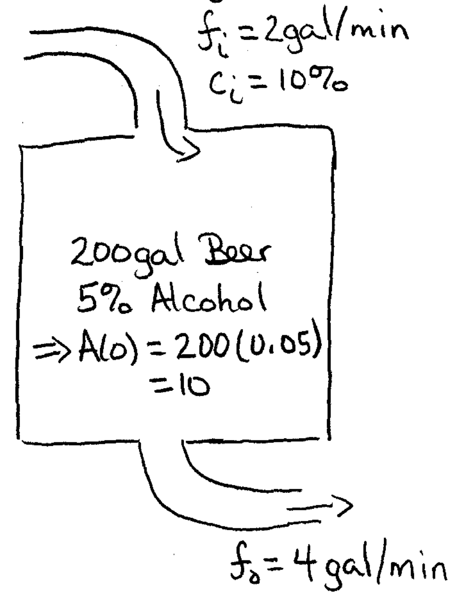
   $x \leq 0,\ y < 0$



B. Without solving it, determine whether you are <u>guaranteed</u> that the differential equation has a unique solution through the given points.

(x_0,y_0)	**Yes/No**
(0,0)	No
(2,3)	Yes
(2,-3)	No
(-1,1)	No

2. Suppose that a large tank holds 200 gal of beer with 5% alcohol. Then another beer with 10% alcohol is pumped into the tank at a rate of 2 gal/min. After the beer is well mixed, it is pumped out at a rate of 4 gal/min. Set up **but not solve** an initial value problem to describe the change in the amount A(t) of alcohol at time t.



$$\frac{dA}{dt} = R_i - R_o$$

time	Vol
0	200
1	200-2
2	200-2(2)
⋮	
t	200-2t

$$R_i = (\text{flow})(\text{concentration})$$
$$= \left(\frac{2 gal}{min}\right)(0.1)$$
$$= 0.2 \ gal/min$$
$$R_o = \left(\frac{4 gal}{min}\right)\left(\frac{A(t)}{200-2t}\right)$$

$$\Rightarrow \begin{cases} \frac{dA}{dt} = 0.2 - \frac{4A}{200-2t} = 0.2 - \frac{2A}{100-t} \\ A(0) = 10 \end{cases}$$

3. Solve $e^x y \frac{dy}{dx} = e^{-y} + e^{-2x-y}$

Note: This is a 1st order, nonlinear, separable equation.

$$e^x y \frac{dy}{dx} = e^{-y}(1+e^{-2x})$$
$$ye^y dy = e^{-x}(1+e^{-2x})\, dx = (e^{-x}+e^{-3x})dx$$
$$\int ye^y dy = \int (e^{-x}+e^{-3x})dx$$
$$u = y \quad du = dy$$
$$dv = e^y \quad v = e^y$$
$$ye^y - \int e^y dy = -e^{-x} - \frac{1}{3}e^{-3x}$$
$$ye^y - e^y = -e^{-x} - \frac{1}{3}e^{-3x} + C$$

4. For the differential equation $\frac{dy}{dx} = (y+1)(y-1)^2$

A. Find the order of the differential equation. Is the differential equation (non)linear, (non)autonomous, and (non)separable? Explain why.

1st order

nonlinear - degree on $y \neq 1$

autonomous - independent variable x does not show up on the right hand side.

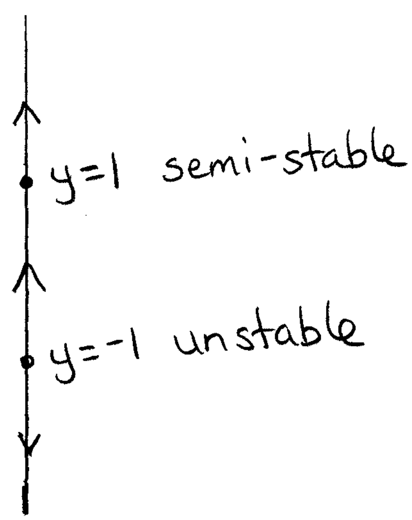
separable - $\frac{dy}{dx} = f(x,y) = g(x)h(y)$ where $g(x) = 1$.

B. Find the critical points for the differential equation. Classify each point as either asymptotically stable, unstable, or semi-stable. Draw the appropriate phase line (portrait).

$$\frac{dy}{dx} = (y+1)(y-1)^2 = 0$$

Critical points: $y = -1, 1$

Interval	Test Value	+/-	↑/↓
$(-\infty, -1)$	-2	$-$	↓
$(-1, 1)$	0	$+$	↑
$(1, \infty)$	2	$+$	↑



5. Consider the differential equation $(x+2)^2 \dfrac{dy}{dx} = 5 - 8y - 4xy$.

A. Find the order of the differential equation. Is the differential equation (non)linear, (non)autonomous, and (non)separable? Explain why.

1st order

linear - DE can rewritten in the form $\frac{dy}{dx} + P(x)y = f(x)$

nonautonomous - x appears explicitly.

nonseparable - cannot be written as $\frac{dy}{dx} = g(x)h(y)$.

B. Give a maximal interval I over which the solution is defined.

$$(x+2)^2\frac{dy}{dx} + 4(x+2)y = 5 \Rightarrow \frac{dy}{dx} + \underbrace{\frac{4}{x+2}}_{P(x)} y = \underbrace{\frac{5}{(x+2)^2}}_{f(x)}$$

$P(x), f(x)$ both cont. when $x \neq -2$
$\Rightarrow I = (-\infty, -2)$ or $(-2, \infty)$

C. Solve the differential equation.

$$\frac{dy}{dx} + \left(\frac{4}{x+2}\right)y = \frac{5}{(x+2)^2}$$

$$IF = e^{\int P(x)dx} = e^{4\int \frac{dx}{x+2}} = e^{4\ln|x+2|} = e^{\ln|(x+2)^4|} = (x+2)^4$$

$$\Rightarrow (x+2)^4\left[\frac{dy}{dx} + \left(\frac{4}{x+2}\right)y = \frac{5}{(x+2)^2}\right]$$

$$(x+2)^4\frac{dy}{dx} + 4(x+2)^3 y = 5(x+2)^2$$

Fail safe: Always check that this is true

$$\frac{d}{dx}\left[(x+2)^4 y\right] = 5(x+2)^2$$

$$\int \frac{d}{dx}\left[(x+2)^4 y\right]dx = 5\int (x+2)^2 dx$$

$$(x+2)^4 y = \frac{5(x+2)^3}{3} + C$$

$$y(x) = \frac{5}{3(x+2)} + \frac{C}{(x+2)^4}$$

Bonus (10 points): A biologist starts with 100 cells in a culture. After 1 day, he has 250 cells. What will be the number of cells after 5 days? Use the differential equation

$$\frac{dP}{dt} = kP$$

where P(t) represents the numbers of cells at time t.

$$\frac{dP}{dt} = KP$$

$$\begin{cases} P(0) = 100 \\ P(1) = 250 \end{cases}$$

<u>Note</u>: This is considered a boundary value problem (BVP) since we are evaluating at more than one value of t.

$$\Rightarrow \frac{dP}{P} = K\,dt$$

$$\int \frac{dP}{P} = K \int dt$$

$$\ln|P| = Kt + C$$

$$P(t) = e^{Kt+C}$$

$$= e^{Kt}e^{C}$$

$$= c_1 e^{Kt}$$

$$P(0) = 100 = c_1 e^{K(0)}$$

$$\Rightarrow c_1 = 100$$

$$\Rightarrow P(t) = 100e^{Kt}$$

$$P(1) = 250 = 100e^{K(1)}$$

$$\Rightarrow \frac{5}{2} = e^{K}$$

$$\Rightarrow K = \ln\left(\frac{5}{2}\right)$$

$$\Rightarrow P(t) = 100e^{\ln\left(\frac{5}{2}\right)t}$$

Now $P(5) = 100\, e^{\ln\left(\frac{5}{2}\right)(5)}$

≈ 9765.63 cells.