MTH 204
Spring 2007
Exam 1

Name: Key

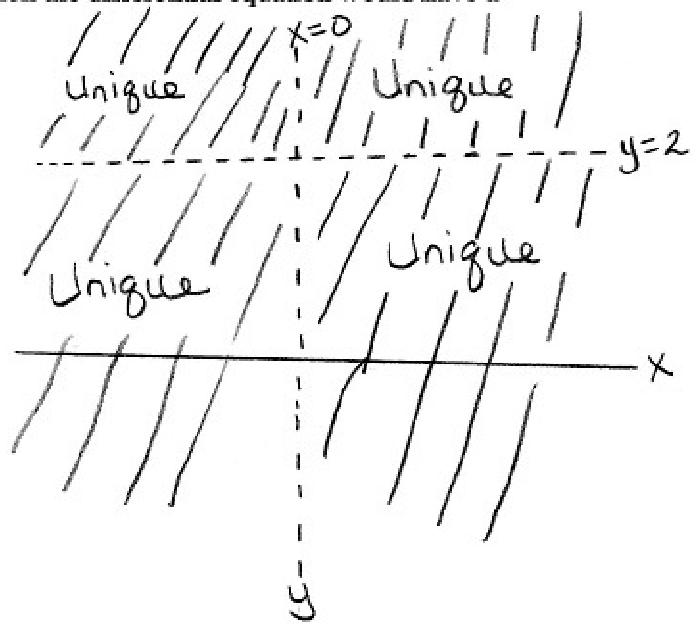
Section: B or C (circle one)

Read the directions carefully.
Each question is worth 20 points.
Write neatly in pencil and **show all your work**
(you will only get credit for what you put on paper).
Please do not share calculators during the test.
If you have trouble during the test, feel free to ask me for help.

Score:________

1. Consider the differential equation $\dfrac{dy}{dx} = f(x,y) = \left(\dfrac{y-2}{x}\right)^{2/3}$.

   A. Determine a region R in the xy-plane for which the differential equation would have a unique solution through each point (x_0,y_0).



1. $f(x,y)$
   $x: x>0, x<0$
   $y: (-\infty,\infty)$

2. $\dfrac{\partial f}{\partial y} = \dfrac{1}{x^{2/3}}\left(\dfrac{2}{3}\right)(y-2)^{-1/3}$
   $= \dfrac{2}{3x^{2/3}(y-2)^{1/3}}$

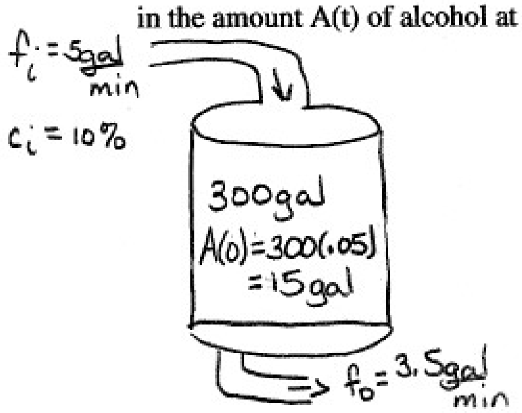
   $x: x>0, x<0$
   $y: y>2, y<2.$

3. region R
   $x: x>0, x<0$
   $y: y>2, y<2$

   B. Without solving it, determine whether you are __guaranteed__ that the differential equation has a unique solution through the given points.

(x_0,y_0)	Yes/No
(0,0)	No
(2,3)	Yes
(-1,2)	No
(-1,1)	Yes

2. Suppose that a large tank holds 300 gal of beer with 5% alcohol. Then another beer with 10% alcohol is pumped into the tank at a rate of 5 gal/min. After the beer is well mixed, it is pumped out at a rate of 3.5 gal/min. Set up **but not solve** an initial value problem to describe the change in the amount A(t) of alcohol at time t.



$$\frac{dA}{dt} = R_i - R_o$$
$$= \left(5\frac{gal}{min}\right)(0.1) - \left(3.5\frac{gal}{min}\right)\left(\frac{A(t)}{300+\frac{3}{2}t}\right)$$
$$= 0.5 - \frac{7A}{600+3t}$$

$A(0) = 15$ gal

t	Vol
0	300
1	$300+5-\frac{7}{2} = 300+\frac{3}{2}$
2	$(300+3/2)+3/2 = 300+3/2(2)$
⋮	
t	$300+\frac{3}{2}t$

A. Find the order of the differential equation. Is the differential equation (non)linear, (non)autonomous, and (non)separable? Explain why.

1st order

linear: can be written in the form $a_1(x)y' + a_0(x)y = g(x)$

nonautonomous: independent variable t is in the equation

nonseparable: $\frac{dA}{dt} \neq g(t)h(A)$

B. What method(s) can you use to solve this equation?

IF method (Variation of parameters)

3. Solve the initial value problem $\frac{dP}{dt} = P - P^2$; $P(0) = \frac{1}{2}$ (an implicit solution is acceptable).

$\frac{dP}{dt} = P - P^2$ Note: This is a 1st order, nonlinear, autonomous separable equation.

$\Rightarrow \frac{dP}{P-P^2} = dt$

Partial Fractions: $\frac{1}{P(1-P)} = \frac{A}{P} + \frac{B}{1-P} = \frac{1}{P} + \frac{1}{1-P}$

$1 = A(1-P) + BP$

$P = 0 \Rightarrow A = 1$

$P = 1 \Rightarrow B = 1$

$\int \frac{dP}{P(1-P)} = \int \left(\frac{1}{P} + \frac{1}{1-P}\right) dP = \int dt$

$\ln|P| - \ln|1-P| = t + C$

$\ln\left|\frac{P}{1-P}\right| = t + C$

$\ln\left|\frac{\frac{1}{2}}{1-\frac{1}{2}}\right| = \ln|1| = 0 = 0 + C$

$\Rightarrow C = 0$

$\Rightarrow \ln\left|\frac{P}{1-P}\right| = t$ implicit solution

$\Rightarrow \frac{P}{1-P} = e^t$

$P = (1-P)e^t = e^t - Pe^t$

$P + Pe^t = P(1+e^t) = e^t$

$\Rightarrow P(t) = \frac{e^t}{1+e^t}$ explicit solution

4. Consider the differential equation $\frac{dy}{dx} = (y-4)^2(y^2-4)$.

A. Find the order of the differential equation. Is the differential equation (non)linear, (non)autonomous, and (non)separable? Explain why.

1st order

nonlinear: equation cannot be written in the form
$a_1(x)y' + a_0(x)y = g(x)$

autonomous: $\frac{dy}{dx} = f(y)$ x does not appear in the equation.

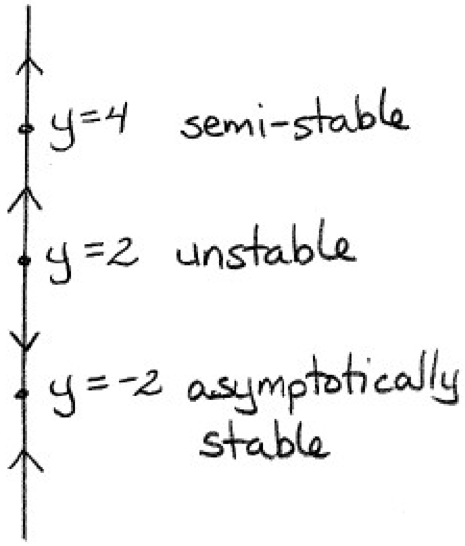
separable: $\frac{dy}{dx} = f(x,y) = g(x)h(y)$ where $g(x) = 1$.

B. Find the critical points for the differential equation. Classify each point as either asymptotically stable, unstable, or semi-stable. Draw the appropriate phase line (portrait).

$\frac{dy}{dx} = (y-4)^2(y^2-4) = 0$

CP: $y = -2, 2, 4$

Int	TV	+/-	↑/↓
$(-\infty,-2)$	-3	+	↑
$(-2,2)$	0	−	↓
$(2,4)$	3	+	↑
$(4,\infty)$	5	+	↑



5. Consider the differential equation $(x+2)^2 \dfrac{dy}{dx} = 5 - 8y - 4xy.$

A. Find the order of the differential equation. Is the differential equation (non)linear, (non)autonomous, and (non)separable? Explain why.

1st order

linear: can be written in the form $a_1(x)y' + a_0(x)y = g(x)$.

nonautonomous: independent variable x is in the equation

nonseparable: $\dfrac{dy}{dx} \neq g(x)h(y)$.

B. Give a maximal interval I over which the solution is defined.

$$\frac{dy}{dx} + \frac{4}{x+2}y = \frac{5}{(x+2)^2} \quad \Rightarrow \quad \begin{array}{l} x>2 \\ x<2 \end{array}$$

C. Solve the differential equation.

$$(x+2)^2\frac{dy}{dx} = 5 - 4(x+2)y \Rightarrow (x+2)^2\frac{dy}{dx} + 4(x+2)y = 5$$

$$\Rightarrow \frac{dy}{dx} + \frac{4}{x+2}y = \frac{5}{(x+2)^2}$$

$$IF = e^{\int P(x)dx} = e^{4\int \frac{dx}{x+2}} = e^{4\ln|x+2|} = e^{\ln|(x+2)^4|} = (x+2)^4$$

$$\Rightarrow (x+2)^4\left[\frac{dy}{dx} + \frac{4}{x+2}y = \frac{5}{(x+2)^2}\right]$$

$$(x+2)^4\frac{dy}{dx} + 4(x+2)^3y = 5(x+2)^2$$

$$\frac{d}{dx}\left[(x+2)^4y\right] = 5(x+2)^2$$

$$\int \frac{d}{dx}\left[(x+2)^4y\right]dx = 5\int (x+2)^2dx$$

$$(x+2)^4y = \frac{5}{3}(x+2)^3 + C$$

$$\Rightarrow y(x) = \frac{5}{3(x+2)} + \frac{C}{(x+2)^4}$$

Bonus (10 points):

A. For what values of r is $y(x) = e^{rx}$ a solution to $3y''+2y'-y=0$?

$y = e^{rx}$

$y' = re^{rx}$

$y'' = r^2e^{rx}$

$3r^2e^{rx}+2re^{rx}-e^{rx}=0$

$e^{rx}(3r^2+2r-1)=0$ $\qquad e^{rx}\neq 0$ for any x

$\Rightarrow 3r^2+2r-1=0$

$\Rightarrow r = \dfrac{-2\pm\sqrt{4-4(3)(-1)}}{2(3)}$

$= \dfrac{-2\pm 4}{6}$

$\Rightarrow r_1 = -1$

$r_2 = \dfrac{1}{3}$

B. For what values of m is $y(x) = x^m$ a solution to $2x^2y''+5xy'-2y=0$?

$y = x^m$

$y' = mx^{m-1}$

$y'' = m(m-1)x^{m-2}$

$2x^2m(m-1)x^{m-2}+5xmx^{m-1}-2x^m=0$

$2m(m-1)x^m+5mx^m-2x^m=0$

$x^m[2m^2-2m+5m-2]=0$ $\qquad x>0$

$2m^2+3m-2=0$

$\Rightarrow m = \dfrac{-3\pm\sqrt{9-4(2)(-2)}}{2(2)}$

$= \dfrac{-3\pm 5}{4}$

$\Rightarrow m_1 = \dfrac{1}{2}$

$m_2 = -2$