MTH 204
Spring 2008
Exam 1 (Sections C & F)

Name: Key

Section: C or F (circle one)

Read the directions carefully.
Each question is worth 20 points,
with a maximum of 100 points possible.
Write neatly in pencil and **show all your work**
(you will only get credit for what you put on paper).
Please do not share calculators during the test.
If you have trouble during the test, feel free to ask me for help.

Score:________

1. Consider the differential equation $\frac{dy}{dx} = (y^2 - 16)(y - 3)^2$.

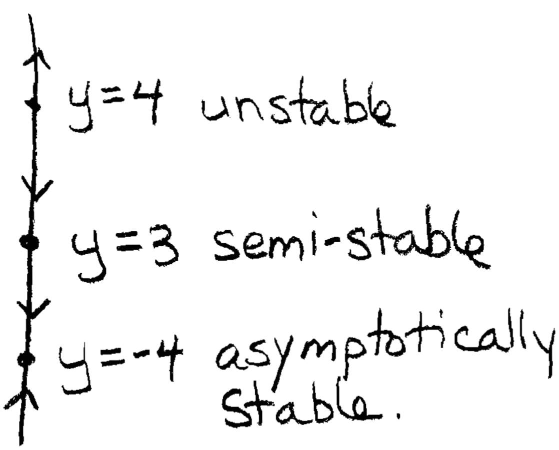
A. Find the order of the differential equation. Is the differential equation (non)linear, (non)autonomous, and (non)separable? Explain why.

1st order
nonlinear: dependent variable is squared
autonomous: independent variable does not show up in the equation
Separable: DE of the form $\frac{dy}{dx} = g(y)h(x)$

B. Find the critical points for the differential equation. Classify each point as either asymptotically stable, unstable, or semi-stable. Draw the appropriate phase portrait.

$\frac{dy}{dx} = (y^2-16)(y-3)^2 = 0 \Rightarrow$ CP: $-4, 3, 4$.

Int	TV	+/-	↗/↙
$(-\infty,-4)$	-5	+	↑
$(-4,3)$	0	−	↓
$(3,4)$	$7/2$	−	↓
$(4,\infty)$	5	+	↑



2. Solve the initial value problem $\frac{dP}{dt} = P - P^2$; $P(0) = \frac{1}{2}$ (an implicit solution is acceptable)

$$\frac{dP}{dt} = P - P^2$$

<u>Note</u>: This is a 1st order, nonlinear, autonomous, <u>separable</u> DE.

$$\Rightarrow \frac{dP}{P(1-P)} = dt$$

Partial Fractions: $\frac{1}{P(1-P)} = \frac{A}{P} + \frac{B}{1-P}$

$$1 = A(1-P) + BP$$

$$P = 0 \Rightarrow A = 1$$

$$P = 1 \Rightarrow B = 1$$

$$\int \frac{dP}{P(1-P)} = \int \left(\frac{1}{P} + \frac{1}{1-P}\right) dP = \int dt$$

$$u = 1 - P$$

$$du = -dP$$

$$\ln|P| - \ln|1-P| = t + C$$

$$\ln\left|\frac{P}{1-P}\right| = t + C$$

$$\ln\left|\frac{1/2}{1-1/2}\right| = \ln|1| = 0 = 0 + C \Rightarrow C = 0$$

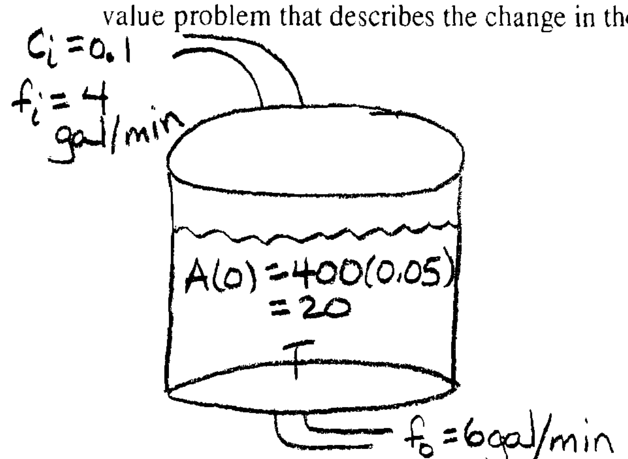
$$\Rightarrow \ln\left|\frac{P}{1-P}\right| = t, \text{ implicit solution}$$

$$\Rightarrow \frac{P}{1-P} = e^t \Rightarrow P = (1-P)e^t = e^t - Pe^t$$

$$P + Pe^t = P(1+e^t) = e^t$$

$$\Rightarrow P(t) = \frac{e^t}{1+e^t}, \text{ explicit solution.}$$

3. Suppose that a large tank (with a capacity of 800 gal) holds 400 gal of beer with 5% alcohol. Then another beer with 10% alcohol is pumped into the tank at a rate of 4 gal/min. After the beer is well mixed, it is pumped out at a rate of 6 gal/min. Set up **but do not solve** an initial value problem that describes the change in the amount A(t) of alcohol at time t.



$$\frac{dA}{dt} = R_i - R_o$$

$$= 4(0.1) - 6\left(\frac{A}{400-2t}\right).$$

$$= 0.4 - \frac{3A}{200-t}$$

$$A(0) = 20$$

Note: ALWAYS write the IC with the DE or else it is not an IVP.

time	Volume
0	400
1	$400+4-6=400-2$
2	$400-2+4-6=400-2(2)$
⋮	
t	$400-2t$

A. Find the order of the differential equation. Is the differential equation (non)linear, (non)autonomous, and (non)separable? Explain why.

1st order

linear; Can be written in the form $A' + p(t)A = f(t)$

nonautonomous; independent variable is in the equation.

nonseparable; cannot be written as $\frac{dA}{dt} = g(A)h(t)$

B. What method(s) can you use to solve this equation?

~~SoV~~ nonseparable

IF linear

C. After how many minutes will the tank be completely filled/emptied?

In 200 minutes, the tank will be empty

4. Consider the differential equation $x^2y' + x(x+2)y = e^x$.

A. Find the order of the differential equation. Is the differential equation (non)linear, (non)autonomous, and (non)separable? Explain why.

1st order

linear: DE is of the form $a_1(x)y' + a_0(x)y = g(x)$

nonautonomous: independent variable is in the equation

nonseparable: DE is not of the form $\frac{dy}{dx} = g(y)h(x)$

B. Give a maximal interval I over which the solution is defined.

Standard form: $y' + \frac{x(x+2)}{x^2}y = \frac{e^x}{x^2} \Rightarrow (-\infty, 0), (0, \infty)$

C. Find an explicit solution to the differential equation.

$$y' + \underbrace{\left(1 + \frac{2}{x}\right)}_{P(x)} y = \frac{e^x}{x^2}$$

$$IF = e^{\int P(x)dx} = e^{\int \left(1+\frac{2}{x}\right)dx} = e^{x + 2\ln x} = x^2 e^x$$

$$x^2e^x\left[y' + \frac{x(x+2)}{x^2}y\right] = x^2e^x\left(\frac{e^x}{x^2}\right) = e^{2x}$$

$$x^2e^xy' + x(x+2)e^xy = e^{2x}$$

$$\frac{d}{dx}[x^2e^xy] = e^{2x}$$

$$\int \frac{d}{dx}[x^2e^xy]dx = \int e^{2x}dx$$

$$x^2e^xy = \frac{1}{2}e^{2x} + C$$

$$y(x) = \frac{e^x}{2x^2} + \frac{C}{x^2e^x}$$

5. Use reduction of order to find a second linearly independent solution to the differential equation $xy''-(x+1)y'+y=0, \quad x>0$, where $y_1(x)=e^x$ is a known solution. (**NO POINTS** for the integral formula).

$y_2(x)=u(x)y_1(x)=ue^x$

$y_2'=u'e^x+ue^x$

$y_2''=u''e^x+2u'e^x+ue^x$

Plug y_2 into DE

$e^x[x(u''+2u'+u)-(x+1)(u'+u)+u]=0$

Regroup in terms of u

$xu''+(2x-x-1)u'+(x-x-1+1)u=0$

$xu''+(x-1)u'=0$

Let $w=u'$, $w'=u''$

$x\frac{dw}{dx}+(x-1)w=0 \Rightarrow \frac{dw}{w}=-\frac{(x-1)}{x}dx$

$\int\frac{dw}{w}=-\int\left(1-\frac{1}{x}\right)dx \Rightarrow \ln|w|=-x+\ln x+K$

$\Rightarrow w=K_1xe^{-x}=u'$

So $u(x)=K_1\int xe^{-x}dx$

$s=x \qquad dt=e^{-x}dx$

$ds=dx \qquad t=-e^{-x}$

$=K_1[-xe^{-x}+\int e^{-x}dx]$.

$=K_1[-x-1]e^{-x}+K_2$ $\qquad$ Pick $\begin{cases}K_1=-1\\K_2=0\end{cases}$

$=(x+1)e^{-x}$

Then $y_2(x)=u(x)y_1(x)$

$=(x+1)e^{-x}e^x$

$=x+1$

Bonus (10 points):

Consider the following differential equations. Along with each equation is a known fact. Give a reason why we may or may not have a fundamental set of solutions.

A. $x^2y''+(x-1)y'+2y=0, \quad x>0$

Known: two solutions on $x>0$.

What can go wrong? Might not be LI

B. $x^2y''-2xy'+y=0, \quad x>0$

Known: two linearly independent functions on $x>0$.

What can go wrong? At least one might not be a solution

C. $13x^3y'''-4x^2y''+4xy'+y=0, \quad x>0$

Known: three linearly independent solutions on $(0,\infty)$.

What can go wrong? Nothing, this is a FSS

D. $x^3y'''-xy'+5y=0, \quad x>0$

Known: three linearly independent solutions on $(-\infty,\infty)$.

What can go wrong? One solution is the absolute value of another

E. $(x-5)y'''+xy'-y=0, \quad x>5$

Known: two linearly independent solutions on $x>5$.

What can go wrong? Not enough solutions