MTH 204
Spring 2009
Exam 1

Name: Key

Section: C or F (circle one)

Read the directions carefully.
Write neatly in pencil and show all your work
(you will only receive credit for what you put on your paper).
Each question is worth 20 points, with
a maximum of 100 points possible.
Please do not share calculators during the test.
DO NO USE decimals on any intermediate step.
If you have trouble during the test, feel free to ask me for help.

Score:_________

1. Consider the differential equation $\dfrac{dy}{dx} = (y-1)^2(y+3)(y-5)$.

A. Find the order of the differential equation. Is the differential equation (non)linear, (non)autonomous, and (non)separable? Explain why.

1st order

nonlinear: coefficient depends on y.

autonomous: x is not explicitly expressed.

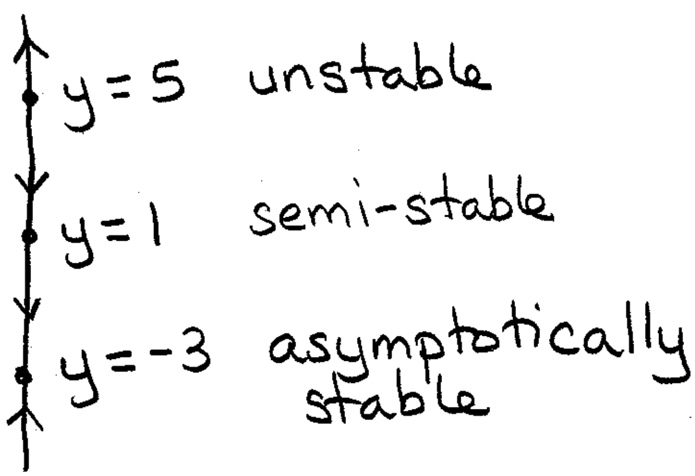
Separable: can written as $\dfrac{dy}{dx} = f(x,y) = g(x)h(y)$

where $g(x) = 1$

B. Find the critical points for the differential equation. Classify each critical point as either asymptotically stable, unstable, or semi-stable. Draw the appropriate phase portrait.

Critical points: $y = -3, 1, 5$

Int	TV	+/-	↑/↓
$(-\infty, -3)$	-4	+	↑
$(-3, 1)$	0	−	↓
$(1, 5)$	2	−	↓
$(5, \infty)$	6	+	↑



2. A model for the population $P(t)$ in a small college is given by the initial value problem

$$\frac{dP}{dt} = P^2 - 4000P, \quad P(0) = 500.$$

1st order, nonlinear, autonomous, Separable

Methods: SoV

A. Solve the IVP. An implicit solution is acceptable.

$$\int \frac{dP}{P(P-4000)} = \int dt$$

$$\frac{1}{P(P-4000)} = \frac{A}{P} + \frac{B}{P-4000} = \frac{\frac{-1}{4000}}{P} + \frac{\frac{1}{4000}}{P-4000}$$

$$1 = A(P-4000) + BP$$

$$P=0 \Rightarrow 1 = -4000A \Rightarrow A = -\frac{1}{4000}$$

$$P=4000 \Rightarrow 1 = 4000B \Rightarrow B = \frac{1}{4000}$$

$$\int\left(\frac{1}{P-4000} - \frac{1}{P}\right)dP = 4000\int dt$$

$$\ln|P-4000| - \ln|P| = \ln\left|\frac{P-4000}{P}\right| = 4000t + C$$

$$\ln\left|\frac{500-4000}{500}\right| = \ln|-7| = 0 + C$$

$$\Rightarrow \ln\left|\frac{P-4000}{P}\right| = 4000t + \ln 7$$

B. At what time will the size of the population be equal to 2000?

Let $P(t_1) = 2000$

$$\ln\left|\frac{-2000}{2000}\right| = 0 = 4000t_1 + \ln 7$$

$$\Rightarrow t_1 = \frac{-\ln 7}{4000} \approx -4.86 \times 10^{-4}$$

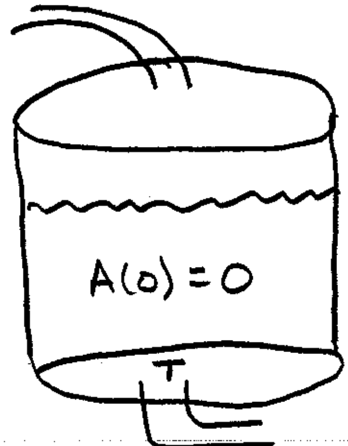
i.e. the population is decreasing.

3. A 800 gallon tank is filled with 600 gallons of pure water. Brine containing 2 lb/gal is pumped into the tank at a rate of 4 gal/min. The well mixed solution is then pumped out at a rate of 6 gal/min. Let $A(t)$ represent the amount of salt in the tank.

A. Set up, but **do not solve** the initial value problem describing the change in the amount of salt in the tank.

$f_i = 4 \frac{gal}{min}$

$c_i = 2 \frac{lb}{gal}$



$f_o = 6\ gal/min$

$$R_i = \left(4 \frac{gal}{min}\right)\left(2 \frac{lb}{gal}\right) = 8 \frac{lb}{min}$$

$$R_o = (6)\left(\frac{A}{600-2t}\right) = \frac{3A}{300-t}$$

$$\begin{cases} \frac{dA}{dt} = R_i - R_o = 8 - \frac{3A}{300-t} \\ A(0) = 0 \end{cases}$$

time	Volume
0	600
1	600+4−6 = 600−2
2	(600−2)+4−6 = 600−2(2)
⋮	
t	600−2t

B. What method(s) can be used to solve this IVP?

1st order, linear, nonautonomous, nonseparable

Methods $\begin{cases} \cancel{\text{SoV}} \\ \text{IF} \end{cases}$

C. At what time is the tank completely empty or filled?

$300 - t = 0$

$\Rightarrow t = 300$

tank is drained after 300 min.

4. Consider the differential equation $xy' - (x-1)y = e^x cos(3x)$.

A. Find the order of the differential equation. Is the differential equation (non)linear, (non)autonomous, and (non)separable? Explain why.

1st order

linear: DE is of the form $a_1(x)y' + a_0(x)y = g(x)$

nonautonomous: x is in the DE.

nonseparable: DE can't be rewritten as

$$\frac{dy}{dx} = f(x,y) = g(x)h(y).$$

B. Give a maximal interval I over which the solution is defined?

$$\begin{cases} x > 0 \\ x < 0 \end{cases}$$

C. Find an explicit solution to the differential equation.

1. Put DE in Std form: $y' - \left(\frac{x-1}{x}\right)y = \frac{e^x \cos(3x)}{x}$

2. Find IF: $IF = e^{-\int(1-\frac{1}{x})dx} = e^{-x+\ln x} = xe^{-x}$

3. Multiply both sides by IF

$$xe^{-x}\left[y' - \left(\frac{x-1}{x}\right)y\right] = xe^{-x}\left[\frac{e^x\cos(3x)}{x}\right]$$

$$xe^{-x}y' - (x-1)e^{-x}y = \cos(3x)$$

4. Rewrite LHS as $\frac{d}{dx}[IF \cdot y]$

$$\frac{d}{dx}\left[xe^{-x}y\right] = \cos(3x)$$

5. Integrate wrt x

$$\int \frac{d}{dx}[xe^{-x}y]dx = \int \cos(3x)dx$$

$$xe^{-x}y = \frac{1}{3}\sin(3x) + C$$

$$\Rightarrow y(x) = \frac{C}{x}e^x + \frac{e^x}{3x}\sin(3x)$$

5. Use reduction of order to find a second linearly independent solution to the differential equation

$$(x-3)y'' - (x-2)y' + y = 0, \quad x > 3,$$

where $y_1(x) = e^x$ is a known solution. (**<u>NO POINTS</u>** will be awarded for the integral formula).

1. Assume $y_2(x) = u(x)y_1(x) = u(x)e^x$ is a solution

$y_2' = u'e^x + ue^x$

$y_2'' = u''e^x + 2u'e^x + ue^x$

2. Plug in y_2

$e^x[(x-3)(u''+2u'+u) - (x-2)(u'+u) + u] = 0$

$(x-3)u'' + (2x-6-x+2)u' + (x-3-x+2+1)u = 0$

$(x-3)u'' + (x-4)u' = 0$

3. Change of variables: Let $\begin{cases} w = u' \\ w' = u'' \end{cases}$

$(x-3)\frac{dw}{dx} + (x-4)w = 0$ 1st order, linear, nonauto, sep,

4. Find w

$\int \frac{dw}{w} = -\int \left(\frac{x-4}{x-3}\right)dx = -\int\left(\frac{x-3-1}{x-3}\right)dx = -\int\left(1 - \frac{1}{x-3}\right)dx$

$\ln|w| = -x + \ln|x-3| + k$

$\Rightarrow w = K_1(x-3)e^{-x} = u'$

5. Find u

$u = K_1\int(x-3)e^{-x}dx$ $\quad s = x-3 \quad dt = e^{-x}dx$
$\quad ds = dx \quad t = -e^{-x}$

$= K_1\left[-(x-3)e^{-x} + \int e^{-x}dx\right]$

$= K_1\left[-xe^{-x} + 3e^{-x} - e^{-x}\right] + K_2$

$= -K_1(x-2)e^{-x} + K_2$ $\quad$ Pick $\begin{cases} K_1 = -1 \\ K_2 = 0 \end{cases}$

$= (x-2)e^{-x}$

6. Find y_2: $y_2 = uy_1 = (x-2)e^{-x}e^x = x-2$

Bonus (10 points): Consider the differential equation $\frac{dy}{dx} = \sqrt{16 - (x^2 + y^2)}$.
Determine a region R in the xy-plane for the differential equation would have a unique solution through each point (x_0, y_0). Sketch the region.

$$\frac{dy}{dx} = \sqrt{16-(x^2+y^2)} = f(x,y)$$

$$\frac{\partial f}{\partial y} = \frac{-y}{\sqrt{16-(x^2+y^2)}}$$

$$R = \{(x,y) : 16-(x^2+y^2) > 0\}$$
$$= \{(x,y) : x^2+y^2 < 16\}$$