MTH 204
Spring 2006
Exam 1

Name: Key

Section: D/E

Read the directions carefully.
Each question is worth 20 points.
Write neatly in pencil and **show all your work**
(you will only get credit for what you put on paper).
Please do not share calculators during the test.
If you have trouble during the test, feel free to ask me for help.

1. Consider the differential equation $\dfrac{dy}{dx} = \sqrt{y^2 - 9} = f(x,y)$.

A. Determine a region R in the xy-plane for the differential equation would have a unique solution through each point (x_0, y_0).

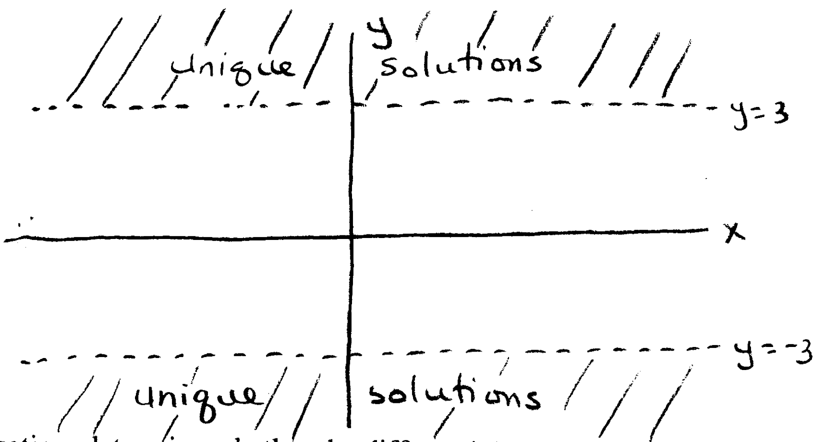
$f(x,y) = \sqrt{y^2-9}$

$\begin{cases} x: (-\infty, \infty) \\ y: (-\infty, -3] \cup [3, \infty) \end{cases}$

$\dfrac{\partial f(x,y)}{\partial y} = \dfrac{d}{dy}\sqrt{y^2-9} = \dfrac{1}{2}(y^2-9)^{-1/2}(y^2)' = \dfrac{y}{\sqrt{y^2-9}}$

$\begin{cases} x: (-\infty, \infty) \\ y: (-\infty, -3) \cup (3, \infty) \end{cases}$

$R \Rightarrow \begin{cases} x: (-\infty, \infty) \\ y > 3, \ y < -3 \end{cases}$



B. Without solving the differential equation, determine whether the differential equation has a unique solution through the given points.

(x_o, y_o)	Yes/No
(1,4)	Yes
(5,3)	No
(2,–3)	No
(–1,1)	No

2. Suppose a cold beer at $40°F$ is placed into a warm room at $70°F$. After 10 minutes, the temperature of the beer is $48°F$. Use Newton's law of cooling to set up, **but not solve**, an "initial value problem" to describe the change in temperature of the beer at time t.

Newton's law of cooling states that the rate of change in the temp of an object is proportional to the difference in temp of the object and the temp of its surrounding medium.

ie $\frac{dT}{dt} = K(T - T_m)$

So $\frac{dT}{dt} = K(T-70)$

$$\begin{cases} T(0) = 40 \\ T(10) = 48. \end{cases}$$

3. Solve $\frac{dy}{dx} = xe^{3x+2y}$ Note: this a 1st order, nonlinear, not autonomous, separable equation.

$$\frac{dy}{dx} = xe^{3x}e^{2y}$$

$$\int e^{-2y}dy = \int \frac{dy}{e^{2y}} = \int xe^{3x}dx$$

Let $u = x$ $du = dx$

$dv = e^{3x}$ $v = \frac{1}{3}e^{3x}$

$$\int e^{-2y}dy = \frac{1}{3}xe^{3x} - \int \frac{1}{3}e^{3x}dx$$

$$-\frac{1}{2}e^{-2y} = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c$$

$$e^{-2y} = -\frac{2}{3}xe^{3x} + \frac{2}{9}e^{3x} + c_1 \qquad c_1 = -2c$$

$$-2y = \ln\left(-\frac{2}{3}xe^{3x} + \frac{2}{9}e^{3x} + c_1\right)$$

So $y(x) = -\frac{1}{2}\ln\left(-\frac{2x}{3}e^{3x} + \frac{2}{9}e^{3x} + c_1\right)$

4. For the differential equation $\dfrac{dy}{dx} = y^2(4 - y^2)$

A. Find the order and state whether the differential equation is linear. If it is nonlinear, explain why. Is the differential equation autonomous or separable? Explain why.

Order: 1st

Nonlinear: y^2 terms.

Autonomous: x is not explicitly expressed (see pt B).

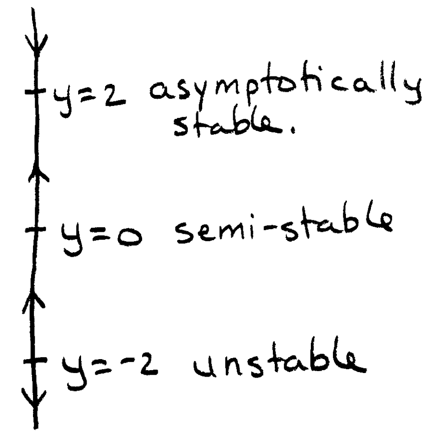
Separable: can be rewritten as $\dfrac{dy}{y^2(4-y^2)} = dx$

B. Find the critical points for the differential equation. Classify each point as either asymptotically stable, unstable, or semi-stable. Draw the appropriate phase line.

$\dfrac{dy}{dx} = y^2(4-y^2) = 0$

CP: $y = -2, 0, 2$

Int	TV	+/-	↑/↓
$(-\infty, -2)$	-3	−	↓
$(-2, 0)$	-1	+	↑
$(0, 2)$	1	+	↑
$(2, \infty)$	3	−	↓



5. Consider the differential equation $x\frac{dy}{dx} - y = x^2 \sin(x)$

A. Find the order and state whether the differential equation is linear. If it is nonlinear, explain why. Is the differential equation autonomous or separable? Explain why.

Order: 1st.

Linear: $\frac{dy}{dx} - \frac{y}{x} = x\sin(x)$: y, y' are of degree 1.

Not autonomous: x appears in the equation.

Not separable: equation is not of the form $\frac{dy}{dx} = g(x)h(y)$

(this implies we need to use the IF).

B. Solve the differential equation and give the largest interval I over which the general solution is defined.

Rewriting the DE in standard form, we have

$$\frac{dy}{dx} \underbrace{-\frac{1}{x}}_{P(x)} y = \underbrace{x\sin(x)}_{Q(x)}$$

$$\text{IF}: e^{\int P(x)\,dx} = e^{-\int \frac{dx}{x}} = e^{-\ln|x|} = e^{\ln|x^{-1}|} = x^{-1}$$

Then $x^{-1}\left[\frac{dy}{dx} - \frac{1}{x}y = x\sin(x)\right]$

$$x^{-1}\frac{dy}{dx} - \frac{1}{x^2}y = \sin(x)$$

$$\frac{d}{dx}\left[x^{-1}y\right] = \sin(x)$$

$$\int \frac{d}{dx}\left[x^{-1}y\right]dx = \int \sin(x)\,dx$$

$$x^{-1}y = -\cos(x) + c$$

$$y(x) = cx - x\cos(x)$$

Then $P(x)$ is cont on $(-\infty, 0)$ and $(0, \infty)$

$Q(x)$ is cont on $(-\infty, \infty)$

So I is where $x > 0$ or $x < 0$.

Bonus (10 points): Find the values of r so that the function $y(x) = e^{rx}$ is a solution to the differential equation $y'' - 5y' + 6y = 0$.

$y(x) = e^{rx}$

$y'(x) = re^{rx}$

$y''(x) = r^2 e^{rx}$

$r^2 e^{rx} - 5r e^{rx} + 6e^{rx} = 0$

$\underbrace{e^{rx}}_{\text{Never zero}}[r^2 - 5r + 6] = 0$

$r^2 - 5r + 6 = 0$

$(r-3)(r-2) = 0$

$r - 3 = 0$ $\qquad$ $r - 2 = 0$

$r_1 = 3$ $\qquad$ $r_2 = 2$

So $y_1(x) = e^{r_1 x} = e^{3x}$

$y_2(x) = e^{r_2 x} = e^{2x}$

We will see this again in section 4.3.