MTH 204
Fall 2008
Exam 2

Name: Key

Section: A or C (circle one)

Read the directions carefully.
Write neatly in pencil and show all your work
(you will only get credit for what you put on paper).
Please do not share calculators during the test.
Each question is worth 20 points
DO NOT USE Decimals on any intermediate step.
The last page contains your Laplace tables.
If you have trouble during the test, feel free to ask me for help.

Score:________

1. Consider the differential equation $x^2y''-2xy'+2y = x^3\cos(x)$.

a. Classify the differential equation by order, linearity, type of coefficients, and state whether or not the equation is homogeneous.

2nd order, linear, variable coefficients, nonhomogeneous
(Cauchy-Euler)

b. What method(s) can you use to solve this equation?

VOP

c. Solve the equation for the interval $x > 0$.

1. Solve $x^2y''-2xy'+2y=0$
Assume $y(x)=x^m \Rightarrow x^m[m^2+(-2-1)m+2]=(m-1)(m-2)=0$
$\Rightarrow m=1,2$
$\Rightarrow y_h(x)=c_1x+c_2x^2$

2. $W(y_1,y_2)=\begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2-x^2=x^2$

3. Std form: $y''-2x^{-1}y'+2x^{-2}y=x\cos(x)$

4. Assume $y_p=u_1y_1+u_2y_2$

"5" Plug in y_p

6. Cramer's Rule

$W_1=\begin{vmatrix} 0 & x^2 \\ x\cos(x) & \underline{} \end{vmatrix} = -x^3\cos(x) \Rightarrow u_1' = \frac{W_1}{W(y_1,y_2)} = -x\cos(x)$

$W_2=\begin{vmatrix} x & 0 \\ \underline{} & x\cos(x) \end{vmatrix} = x^2\cos(x) \Rightarrow u_2' = \frac{W_2}{W(y_1,y_2)} = \cos(x)$

7. Integrate & plug in

$u_1(x) = -\int x\cos(x)dx = -x\sin(x)+\int\sin(x)dx$
$= -x\sin(x)-\cos(x)+\cancel{K_1}$

$u_2(x)=\int\cos(x)dx = \sin(x)+\cancel{K_2}$

$\Rightarrow y_p = -(x\sin(x)+\cos(x))x+(\sin(x))x^2 = -x\cos(x)$

no terms absorbed

8. GS: $y(x)=y_h+y_p$
$=c_1x+c_2x^2-x\cos(x)$

2. A spring with a 4 kilogram object hangs vertically at equilibrium. A force of 40 Newtons applied to the spring is known to stretch it 2 meters. The surrounding medium exerts a damping force proportional to the velocity of a body moving through it, and it is known that a velocity of 4 meters per second results in a damping force of 36 Newtons. At t = 0, the object is 3 meters above equilibrium position and then released with an downward velocity of 2 meters per second. Also, assume there is an external force acting on the spring given by $f(t) = 12\sin(\gamma t)$.

a. Set up, **do not solve**, the IVP describing this motion.

$m = 4$

$F_R = ks \Rightarrow 40 = 2k$
$\Rightarrow k = 20$

$F_D = \beta v \Rightarrow 36 = 4\beta$
$\Rightarrow \beta = 9$

$$4y'' + 9y' + 20y = 12\sin(\gamma t)$$
$$\begin{cases} y(0) = -3 \\ y'(0) = 2 \end{cases}$$

b. Rewriting the above spring-mass problem as a circuit problem, give the inductance, resistance, capacitance, and impressed voltage. Make sure you carefully label each term.

$$Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$L = 4\ h$ $\qquad E(t) = 12\sin(\gamma t)\ V$

$R = 9\,\Omega$

$C = \frac{1}{20}\ f$

c. Now remove the damping force. For what value of γ does resonance occur?.

Resonance: $\omega = \gamma$

$4y'' + 20y = 0 \Rightarrow y'' + 5y = 0$, $y(t) = e^{rt}$
$\Rightarrow r^2 + 5 = 0$
$\Rightarrow r = \pm\sqrt{5}\,i$
$\Rightarrow \gamma = \sqrt{5}$

3. Consider the differential equation $y''' - 8y = 2e^{2t} + 3e^{t}$.

a. Classify the differential equation by order, linearity, type of coefficients, and state whether or not the equation is homogeneous.

3rd order, linear, constant coefficients, nonhomogeneous

b. What method(s) can you use to solve this equation? What are the potential drawbacks to each method?

MUC - annihilator/product rule
VOP - 3x3 Wronskian
Laplace - Must fudge ICs / Messy partial fractions

c. Find a particular solution (you may find the identity $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ useful).

1. Solve $y''' - 8y = 0$, $y(t) = e^{rt}$

$$\Rightarrow \cancel{e^{rt}}\,[r^3 - 8] = 0 \Rightarrow (r-2)(r^2+2r+4) = 0$$

$$r = \frac{-2 \pm \sqrt{4-4(1)(4)}}{2} = \frac{-2 \pm 2\sqrt{3}\,i}{2} = -1 \pm \sqrt{3}\,i,\ r = 2$$

$$\Rightarrow y_h(t) = c_1 e^{2t} + e^{-t}[c_2\cos(\sqrt{3}t) + c_3\sin(\sqrt{3}t)]$$

2. Roots of $g(t)$: $r = 2, 1$
3. Char Eq for $g(t)$: $(r-2)(r-1) = 0$
4. Ann: $(D-2)(D-1)$
5. $(D-2)(D-1)(D^3-8)y = (D-2)(D-1)g(t) = 0$
6. $y(t) = e^{rt} \Rightarrow (r-2)(r-1)(r^3-8) = 0$

$$\Rightarrow r = 2, -1 \pm \sqrt{3}\,i, 1, 2$$

$$\Rightarrow y(t) = \underbrace{c_1 e^{2t} + e^{-t}[c_2\cos(\sqrt{3}t) + c_3\sin(\sqrt{3}t)]}_{y_h} + \underbrace{c_4 e^{t} + c_5 te^{2t}}_{y_p}$$

7. $y_p = Ae^t + Bte^{2t}$ $\qquad y_p'' = Ae^t + 4Be^{2t} + 4Bte^{2t}$

$y_p' = Ae^t + Be^{2t} + 2Bte^{2t}$ $\qquad y_p''' = Ae^t + 12Be^{2t} + 8Bte^{2t}$

8. MUC: $y_p''' - 8y_p = (A-8A)e^t + \cancel{(8B-8B)}te^{2t} + 12Be^{2t}$

$$= -7Ae^t + 12Be^{2t} = 2e^{2t} + 3e^t$$

$$\Rightarrow A = -\frac{3}{7},\ B = \frac{1}{6}$$

$$\Rightarrow y_p(t) = -\frac{3}{7}e^t + \frac{1}{6}te^{2t}$$

4. Find the inverse Laplace transform of the following:

a. $F(s) = \left\{ \dfrac{(s-2)^2}{s^4} \right\}$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{ \frac{s^2-4s+4}{s^4} \right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{2!}{s^{2+1}}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{3!}{s^{3+1}}\right\}$$

$$= t - 2t^2 + \frac{2}{3}t^3$$

b. $G(s) = \dfrac{s-6}{s[(s-1)^2+2]} = \dfrac{A}{s} + \dfrac{Bs+C}{(s-1)^2+2} = \dfrac{-2}{s} + \dfrac{2s-3}{(s-1)^2+2}$

$$s-6 = A[(s-1)^2+2] + Bs^2 + Cs$$

$s=0 \Rightarrow -6 = 3A \Rightarrow A=-2$

$s=1 \Rightarrow -5 = -2(2) + B + C \quad \Rightarrow B+C=-1$

$s=-1 \Rightarrow -7 = -2(6) + B - C \quad \Rightarrow \underline{B-C=5}$

$2B = 4 \Rightarrow B=2,\ C=-3$

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = -2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{2s-2-1}{(s-1)^2+2}\right\}$$

$$= -2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 2\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+(\sqrt{2})^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+(\sqrt{2})^2}\right\}$$

$$= -2 + 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+(\sqrt{2})^2}\right\}_{s \to s-1} - \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+(\sqrt{2})^2}\right\}_{s \to s-1}$$

$$= -2 + 2e^t\cos(\sqrt{2}t) - \frac{1}{\sqrt{2}}e^t\sin(\sqrt{2}t)$$

5. Consider the IVP $y''+y=\delta\left(t-\frac{\pi}{2}\right)+\delta(t-\pi)$ subject to $y(0)=1$ and $y'(0)=0$.

a. Solve the IVP.

1. Take the Laplace

$$\mathcal{L}\{y''\}+\mathcal{L}\{y\}=\mathcal{L}\{\delta(t-\tfrac{\pi}{2})\}+\mathcal{L}\{\delta(t-\pi)\}$$

$$(s^2Y(s)-s\overset{=1}{y(0)}-\overset{=0}{y'(0)})+Y(s)=e^{-\frac{\pi}{2}s}+e^{-\pi s}$$

$$(s^2+1)Y(s)=s+e^{-\frac{\pi}{2}s}+e^{-\pi s}$$

2. Find $Y(s)$

$$Y(s)=\frac{s}{s^2+1}+\frac{e^{-\frac{\pi}{2}s}}{s^2+1}+\frac{e^{-\pi s}}{s^2+1}=\frac{s}{s^2+1}+e^{-\frac{\pi}{2}s}F(s)+e^{-\pi s}F(s)$$

where $F(s)=\frac{1}{s^2+1}$

3. Find $f(t)$

$$f(t)=\mathcal{L}^{-1}\{F(s)\}=\sin(t)$$

4. Find $y(t)$

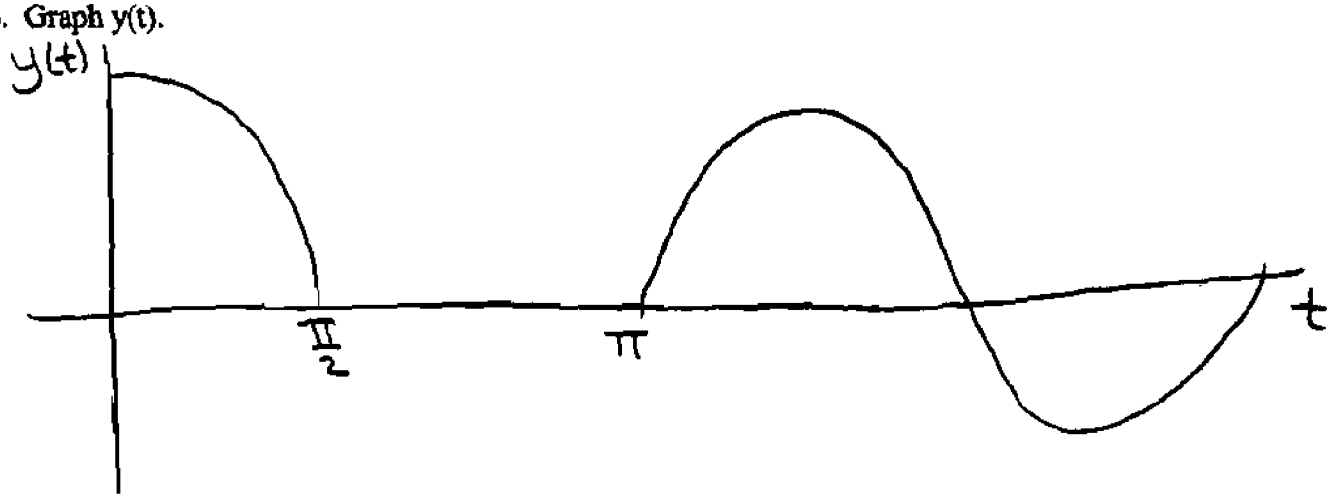
$$y(t)=\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}+\mathcal{L}^{-1}\{e^{-\frac{\pi}{2}s}F(s)\}+\mathcal{L}^{-1}\{e^{-\pi s}F(s)\}$$

$$=\cos(t)+f(t-\tfrac{\pi}{2})\mathcal{U}(t-\tfrac{\pi}{2})+f(t-\pi)\mathcal{U}(t-\pi)$$

$$=\cos(t)-\cos(t)\mathcal{U}(t-\tfrac{\pi}{2})-\sin(t)\mathcal{U}(t-\pi)$$

$$=\begin{cases}\cos(t), & 0\le t<\frac{\pi}{2}\\ \cos(t)-\cos(t)=0, & \frac{\pi}{2}\le t<\pi\\ \cos(t)-\cos(t)-\sin(t)=-\sin(t), & t\ge\pi\end{cases}$$

b. Graph $y(t)$.



Bonus (10 points):

a. Find the annihilator of the smallest order for the function

$$g(x) = 3x^2 - 14xe^{2x} + 10\cos(2x) + e^x \sin(x).$$

Roots: $r = 0, 0, 0, 2, 2, \pm 2i, 1 \pm i$

Char Eq: $r^3(r-2)^2(r^2+4)[(r-1)^2+1] = 0$

Ann: $D^3(D-2)^2(D^2+4)[(D-1)^2+1]$

b. Use the Heaviside function, $\mathcal{U}(t-a)$, to rewrite the piecewise function

$$f(t) = \begin{cases} t, & 0 \le t < 1 \\ t+1, & 1 \le t < 2 \\ 2 & t \ge 2 \end{cases}$$

in a more compact form.

$$f(t) = t[\underbrace{\mathcal{U}(t-0)}_{1} - \mathcal{U}(t-1)] + (t+1)[\mathcal{U}(t-1) - \mathcal{U}(t-2)] + 2\,\mathcal{U}(t-2)$$

$$= t + (-t + t + 1)\mathcal{U}(t-1) + (-t - 1 + 2)\mathcal{U}(t-2)$$

$$= t + \mathcal{U}(t-1) - (t-1)\mathcal{U}(t-2)$$