MTH 204
Fall 2005
Exam 2

Name: Key

Section: A/C

Math 204
Sections A & C (Wintz)
Exam 2
Fall 2005

Read the directions carefully.
Each question is worth 20 points.
Write neatly in pencil and **show all your work**
(you will only get credit for what you put on paper).
DO NOT use decimals in any intermediate step
Please do not share calculators during the test.
If you have trouble during the test, feel free to ask me for help.

Note: when we rewrite the DE in standard form:
$y'' - \frac{2}{x}y' - \frac{4}{x^2}y = 0$
we can see that the largest intervals sols of this DE can exist are $(-\infty, 0)$ and $(0, \infty)$

Score:__________

1. Consider the differential equation $x^2y''-2xy'-4y = 0$.

   a. Classify the differential equation by order, linearity, and whether the equation is homogenous.

   2nd, linear, homogenous } 1 pt each

   For future reference, this called a Cauchy-Euler equation.

   b. What does it mean to be a fundamental set of solutions?

   1. $y_1, y_2, \dots, y_n$ are sols to the DE.
   2. $y_1, y_2, \dots, y_n$ are Linearly independent.
   3. Number of solutions = order of DE.

   } 1 pts each

   c. Let $f(x) = x^{-1}$, $g(x) = x^{-2}$, and $h(x) = x^4$. Do f(x) and g(x) form a fundamental set of solutions for the differential equation? Do f(x) and h(x)?

2pts each

$f(x) = x^{-1}$, $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$
$x^2(2x^{-3}) - 2x(-x^{-2}) - 4x^{-1} = 2x^{-1} + 2x^{-1} - 4x^{-1} = 0 \Rightarrow f(x)$ is a sol.

$g(x) = x^{-2}$, $g'(x) = -2x^{-3}$, $g''(x) = 6x^{-4}$
$x^2(6x^{-4}) - 2x(-2x^{-3}) - 4x^{-2} = 6x^{-2} + 4x^{-2} - 4x^{-2} = 6x^{-2}$
$\Rightarrow g(x)$ is not a sol

$h(x) = x^4$, $h'(x) = 4x^3$, $h''(x) = 12x^2$
$x^2(12x^2) - 2x(4x^3) - 4x^4 = 12x^4 - 8x^4 - 4x^4 = 0 \Rightarrow h(x)$ is a sol.

2pt { Since g(x) is not a sol, f(x) and g(x) do not form a FSS.

3pt

$$W(f,h)(x) = \begin{vmatrix} x^{-1} & x^4 \\ -x^{-2} & 4x^3 \end{vmatrix} = (x^{-1})(4x^3) - (-x^{-2})(x^4)$$
$$= 4x^2 + x^2$$
$$= 5x^2$$
$$\neq 0 \quad \text{since } x \neq 0.$$

1pt { $\Rightarrow f(x)$ and $h(x)$ are linearly independent.

2pt { Since f(x) and h(x) are linearly independent solutions of $x^2y'' - 2xy' - 4y = 0$ (note the number of solutions matches the order of the DE), they form a FSS.

2. A tank contains 200 liters of fluid in which 30 grams of salt is dissolved. Suppose that pure water is then pumped into the tank a rate of 2 L/min; the well-mixed solution is then pumped out at 3 L/min. Find the amount A(t) of grams of salt at time t.

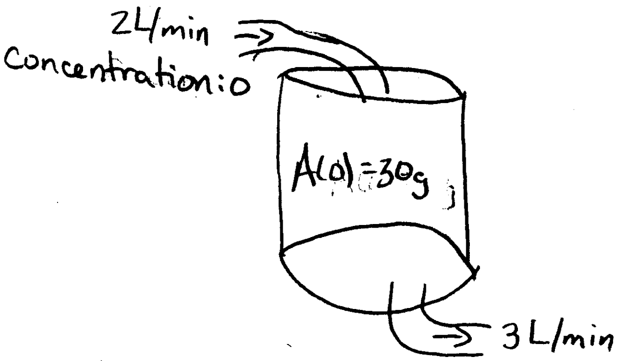
$$\frac{dA}{dt} = R_i - R_o$$

$$R_i = \text{concentration} \cdot \text{flow} = 0 \cdot \left(\frac{2L}{min}\right) = 0$$

$$R_o = \text{''} \quad \text{''} = \left(\frac{A(t)}{200-t} \frac{g}{L}\right)\left(\frac{3L}{min}\right) = \frac{3A}{200-t}$$

Note: we are losing a liter per min.

5pts



So the IVP is

$$\begin{cases} \dfrac{dA}{dt} = 0 - \dfrac{3A}{200-t} \\ A(0) = 50 \end{cases}$$

5pts
$$\frac{dA}{dt} = \frac{-3A}{200-t}$$
$$\int \frac{dA}{A} = -3\int \frac{dt}{200-t}$$

5pts
$$\ln|A| = -3(-\ln|200-t|) + c$$
$$\ln|A| = 3\ln|200-t| + c$$
$$\ln|A| = \ln|(200-t)^3| + c$$

2pts $A(t) = e^{\ln|(200-t)^3| + c} = e^{\ln|(200-t)^3|} e^{c} = c_1(200-t)^3$

2pts
$$A(0) = 30 = c_1(200-0)^3$$
$$\Rightarrow c_1 = \frac{30}{200^3} = \frac{3}{800{,}000} \approx 3.75E-6$$

1pt. $A(t) = (3.75E-6)(200-t)^3$

3. Solve the initial value problem $y'''+2y''+y'+2y=0$, $y(0)=1, y'(0)=1, y''(0)=0$.

5pts.
$$y(x)=e^{rx}$$
$$y'(x)=re^{rx}$$
$$y''(x)=r^2e^{rx}$$
$$y'''(x)=r^3e^{rx}$$
$$r^3e^{rx}+2r^2e^{rx}+re^{rx}+2e^{rx}=0$$
$$e^{rx}(r^3+2r^2+r+2)=0.$$

5pts
$$r^3+2r^2+r+2=0$$
$$r^2(r+2)+(r+2)=0$$
$$(r^2+1)(r+2)=0$$
$$r^2+1=0 \qquad r+2=0$$
$$r=\pm i \qquad r=-2$$

4pts
$$y(x)=c_1\cos(x)+c_2\sin(x)+c_3e^{-2x}$$
$$y'(x)=-c_1\sin(x)+c_2\cos(x)-2c_3e^{-2x}$$
$$y''(x)=-c_1\cos(x)-c_2\sin(x)+4c_3e^{-2x}$$

5pts
$$y(0)=1=c_1+c_3$$
$$y'(0)=1=c_2-2c_3$$
$$y''(0)=0=-c_1+4c_3 \Rightarrow c_1=4c_3$$
$$1=c_1+c_3=4c_3+c_3=5c_3 \Rightarrow c_3=\frac{1}{5}$$
$$\Rightarrow c_1=\frac{4}{5}$$
$$1=c_2-2c_3=c_2-\frac{2}{5} \Rightarrow c_2=\frac{7}{5}$$

1pt.
So $y(x)=\frac{4}{5}\cos(x)+\frac{7}{5}\sin(x)+\frac{1}{5}e^{-2x}$

4. **Use reduction of order to find a second linearly independent solution to the differential equation** $x^2y''+3xy'+y=0,\ \ x>0;\ \ y_1(x)=x^{-1}$. **(Do not use the integration formula)**

5pts
$$y_2 = uy_1 = ux^{-1}$$
$$y_2' = u'x^{-1} - ux^{-2}$$
$$y_2'' = u''x^{-1} - u'x^{-2} - u'x^{-2} + 2ux^{-3} = u''x^{-1} - 2u'x^{-2} + 2ux^{-3}$$

5pts
$$x^2(u''x^{-1} - 2u'x^{-2} + 2ux^{-3}) + 3x(u'x^{-1} - ux^{-2}) + ux^{-1} = 0$$
$$u''x - 2u' + 2ux^{-1} + 3u' - 3ux^{-1} + ux^{-1} = 0$$
$$u''x + u'(-2+3) + u(\cancel{2x^{-1} - 3x^{-1} + x^{-1}}) = 0$$
$$u''x + u' = 0$$

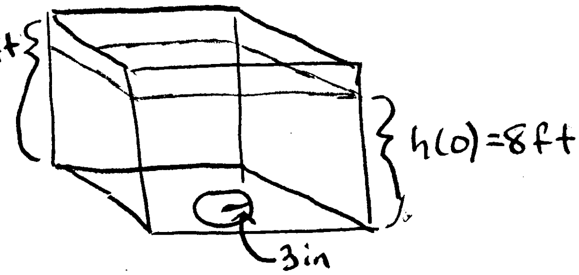
1pt.
let $w = u'$
$w' = u''$

4pts
$$x\frac{dw}{dx} + w = 0$$
$$x\frac{dw}{dx} = -w$$
$$\int \frac{dw}{w} = -\int \frac{dx}{x}$$
$$\ln|w| = -\ln|x| + c = \ln|x^{-1}| + c$$
$$w = e^{\ln|x^{-1}|+c} = c_1x^{-1} = u'$$

4pts
$$\frac{du}{dx} = c_1x^{-1}$$
$$\int du = c_1\int x^{-1}dx$$
$$u = c_1\ln|x| + c_2 \qquad \text{let } \begin{cases} c_1 = 1 \\ c_2 = 0 \end{cases}$$
$$u = \ln|x|$$

1pt So $y_2 = u(x)\,y_1(x) = x^{-1}\ln|x|$.

5. Suppose that a **cube-shaped** tank is leaking water through a circular hole its bottom. When friction and contraction are ignored, the height h is described by $\frac{dh}{dt} = \frac{-A_h}{A_w}\sqrt{2gh}$ where A_h and A_w represent the cross-sectional areas of the hole and water, respectively. Suppose that the length of the tank is 10 ft and the radius of the hole is 3 inches. Find the height of the water, h at time t. If the tank is initially 80% full, how long will it take for the tank to completely empty? Use $g = 32\text{ft}/\text{s}^2$.



4pts
$$A_h = \pi r^2 = \pi\left(\frac{3}{12}\right)^2 = \frac{\pi}{16}\ \text{ft}^2$$
$$A_w = lw = 10^2 = 100\ \text{ft}^2$$

$$\frac{dh}{dt} = -\frac{\frac{\pi}{16}}{100}\sqrt{2(32)h}$$
$$h(0) = 8.$$

5pts
$$\frac{dh}{dt} = \frac{-\frac{\pi}{16}}{100} 8\sqrt{h} = \frac{-\pi}{200}\sqrt{h}$$
$$\int \frac{dh}{\sqrt{h}} = \frac{-\pi}{200}\int dt$$

6pts
$$2\sqrt{h} = \frac{-\pi}{200}t + c$$
$$\sqrt{h} = \frac{-\pi}{400}t + c_1$$
$$h(t) = \left(\frac{-\pi}{400}t + c_1\right)^2$$

2pts
$$h(0) = 8 = (0 + c_1)^2 \Rightarrow c_1 = \sqrt{8} = 2\sqrt{2}$$
$$\text{So } h(t) = \left(\frac{-\pi}{400}t + 2\sqrt{2}\right)^2$$

3pts
$$\text{Then } 0 = \left(\frac{-\pi}{400}t + 2\sqrt{2}\right)^2$$
$$0 = \frac{-\pi}{400}t + 2\sqrt{2}$$
$$\frac{\pi}{400}t = 2\sqrt{2}$$
$$t = (2\sqrt{2})\left(\frac{400}{\pi}\right) = \frac{800\sqrt{2}}{\pi} \approx 360.13\text{s or } 6\text{ mins.}$$