**MTH 204**
**Fall 2006**
**Exam 2**

**Name:** Key
**Section:** B

**Read the directions carefully.**
**Each question is worth 20 points.**
**Write <u>neatly</u> in pencil and <u>show all your work</u>**
**<u>(you will only get credit for what you put on paper)</u>.**
**<u>Do not</u> use decimals in any intermediate step.**
**Please do not share calculators during the test.**
**If you have trouble during the test, feel free to ask me for help.**

1. Solve the differential equation $y^{(4)} - y''' - y' + y = 0$.

Assume $y(x) = e^{rx}$

$$y' = re^{rx}$$
$$y'' = r^2 e^{rx}$$
$$y''' = r^3 e^{rx}$$
$$y^{iv} = r^4 e^{rx}$$

$$\Rightarrow r^4 e^{rx} - r^3 e^{rx} - re^{rx} + e^{rx} = 0$$
$$e^{rx}(r^4 - r^3 - r + 1) = 0$$
$$\Rightarrow r^4 - r^3 - r + 1 = 0$$
$$r^3(r-1) - (r-1) = 0$$
$$(r-1)(r^3-1) = 0$$
$$(r-1)(r-1)(r^2+r+1) = 0$$

$$\Rightarrow r = 1, 1 \qquad r^2 + r + 1 = 0$$
$$r = \frac{-1 \pm \sqrt{1 - 4(1)(1)}}{2(1)} = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$y_1 = e^x$$
$$y_2 = xe^x$$
$$y_3 = e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right)$$
$$y_4 = e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$y(x) = c_1 e^x + c_2 xe^x + e^{-\frac{1}{2}x}\left[c_3 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_4 \sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

2. Consider the differential equation $x^2y''-6xy'+12y=0$ on the interval $(0,\infty)$.

A. Give the three conditions needed to have a fundamental set of solutions to this equation on $(0,\infty)$.

i. $y_1, y_2$ are solutions to the DE
ii. $y_1, y_2$ are linearly independent
iii. # linearly independent solutions = order of DE

B. Consider the functions $f(x)=3x^4$, $g(x)=3x^2$, and $h(x)=x^4$. Do f(x) and g(x) form a fundamental set of solutions? Why or why not? Do f(x) and h(x) form a fundamental set of solutions? Why or why not?

$f=3x^4 \quad f'=12x^3 \quad f''=36x^2$

$x^2f''-6xf'+12f=x^2(36x^2)-6x(12x^3)+12(3x^4)=36x^4-72x^4+36x^4=0$

$\Rightarrow f$ is a solution

$g=3x^2 \quad g'=6x \quad g''=6$

$x^2g''-6xg'+12g=x^2(6)-6x(6x)+12(3x^2)=6x^2-36x^2+36x^2\neq 0$

$\Rightarrow g$ is not a solution

$\Rightarrow f,g$ cannot form a FSS since g is not a solution

By the Superposition Principle, since f is a solution h is a solution as well. However since f, h are scalar multiples of each other, they do not form a FSS.

3. $y_1(x) = x^4$ is a solution to the differential equation $x^2y''-7xy'+16y=0$. Use reduction of order to find a second linearly independent solution on the interval $(0,\infty)$. No points will be awarded if you use the integral formula.

$y_2(x) = u(x)\,y_1(x) = ux^4$

$y_2' = u'x^4 + 4ux^3$

$y_2'' = u''x^4 + 4u'x^3 + 4u'x^3 + 12ux^2 = u''x^4 + 8u'x^3 + 12ux^2$

$x^2(u''x^4 + 8u'x^3 + 12ux^2) - 7x(u'x^4 + 4ux^3) + 16ux^4 = 0$

Regroup in terms of $u$

$x^6u'' + \underbrace{(8x^5 - 7x^5)}_{x^5}u' + \underbrace{(12x^4 - 28x^4 + 16x^4)}_{0}u = 0$

$x^6u'' + x^5u' = 0$ Change of Variables $\begin{cases} w = u' \\ w' = u'' \end{cases}$

$\Rightarrow x^6\dfrac{dw}{dx} + x^5w = 0$

Note: This is a 1st order, linear, separable equation

$\Rightarrow \displaystyle\int \frac{dw}{w} = -\int \frac{dx}{x}$

$\ln|w| = -\ln|x| = \ln|x^{-1}|$

$w = x^{-1} = u'$

$\Rightarrow u = \displaystyle\int \frac{dx}{x} = \ln x$

$\Rightarrow y_2 = uy_1 = x^4 \ln x$

4. Two chemicals A and B are combined to form a chemical C. The rate of the reaction is proportional to the product of the instantaneous amounts of A and B not converted to C. Initially, there are 40 grams of A and B each, and for every 2 grams of B, 1 gram of A is used. It is observed after 10 minutes that 5 grams of C are formed.

A. Set up the differential equation with the appropriate "boundary" conditions.

Let $X(t)$ be amount of C at time t.

remaining A: $40-\frac{1}{1+2}X = 40-\frac{1}{3}X$

remaining B: $40-\frac{2}{1+3} = 40-\frac{2}{3}X$

$$\Rightarrow \frac{dX}{dt} = k\left(40-\frac{X}{3}\right)\left(40-\frac{2X}{3}\right) \qquad \begin{cases} X(0)=0 \\ X(10)=5 \end{cases}$$

B. Give the order of the differential equation and state whether the equation is linear or nonlinear, separable or non-separable, and autonomous or non-autonomous.

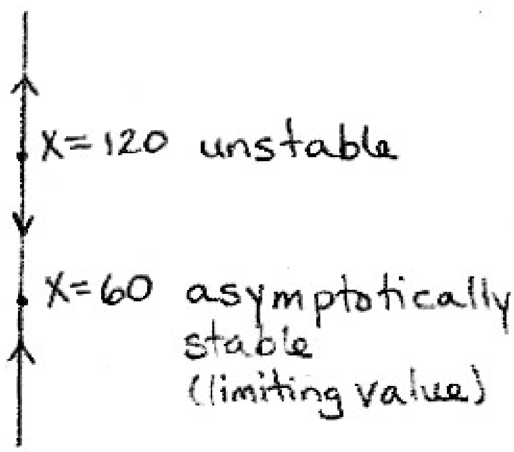
1st order, nonlinear, separable, autonomous

C. What is the limiting amount of C after a long time? How much of A and B are left over a long time? (Hint: What are the critical points?; **DO NOT** solve the DE)

$$\frac{dX}{dt} = k\left(40-\frac{X}{3}\right)\left(40-\frac{2X}{3}\right) = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)k(120-X)(60-X) = k_1(120-X)(60-X) = 0$$

Critical pts: $X = 60, 120$

| Interval | TV | +/- | ↑/↓ |
|---|---|---|---|
| $(-\infty, 60)$ | 0 | + | ↑ |
| $(60, 120)$ | 100 | − | ↓ |
| $(120, \infty)$ | 130 | + | ↑ |




Limiting value of C after a long time: 60g.

A remaining: $40-\frac{1}{3}(60) = 20g$

B remaining: $40-\frac{2}{3}(60) = 0g$

5. Find a particular solution $y_p$ to the differential equation $y''-9y=4e^{3x}+2x$ using the method of undetermined coefficients.

Solve for $y''-9y=0$

Assume $y=e^{rx}$

$\Rightarrow r^2-9=0$

$(r-3)(r+3)=0, \quad r=\pm 3$

$\Rightarrow y_h=c_1e^{-3x}+c_2e^{3x}$

$g(x)=4e^{3x}+2x \quad r=3,0,0$

CE: $r^2(r-3)=0$

Ann: $D^2(D-3)$

$\Rightarrow D^2(D-3)(D^2-9)y=D^2(D-3)[4e^{3x}+2x]=0$

Assume $y=e^{rx}$

$\Rightarrow r^2(r-3)(r-3)(r+3)=0, \quad r=-3,3,3,0,0$

$\Rightarrow y(x)=\underbrace{c_1e^{-3x}+c_2e^{3x}}_{y_h}+\underbrace{c_3xe^{3x}+c_5+c_6x}_{y_p}$

$\Rightarrow y_p=Axe^{3x}+B+Cx$

$y_p'=Ae^{3x}+3Axe^{3x}+C$

$y_p''=6Ae^{3x}+9Axe^{3x}$

$\Rightarrow y_p''-9y_p=(9A-9A)xe^{3x}+6Ae^{3x}-9B-9Cx$

$=6Ae^{3x}-9B-9Cx$

$=4e^{3x}+2x$

$\Rightarrow 6A=4$

$-9B=0$

$-9C=2$

$\Rightarrow A=\frac{2}{3}, \quad B=0, \quad C=\frac{-2}{9}$

$y_p=\frac{2}{3}xe^{3x}-\frac{2}{9}x$

**Bonus** (10 points): For what values of m is $y(x) = x^m$ a solution to the differential equation

$x^2y''+7xy'+8y=0$?

$y = x^m$
$y' = mx^{m-1}$
$y'' = m(m-1)x^{m-2}$

$x^2[m(m-1)x^{m-2}]+7x[mx^{m-1}]+8x^m=0$
$m(m-1)x^m+7mx^m+8x^m=0$
$x^m[m^2-m+7m+8]=0 \quad , x>0$
$\Rightarrow m^2+6m+8=0$
$(m+2)(m+4)=0$
$m_1=-2$
$m_2=-4$
$y_1=x^{-2}$
$y_2=x^{-4}$