Read the directions carefully.
Each question is worth 20 points.
Write <u>neatly</u> in pencil and **show all your work**
(you will only get credit for what you put on paper).
Please do not share calculators during the test.
If you have trouble during the test, feel free to ask me for help.

Score: _____

1. Consider the following differential equations. Along with each equation is a known fact. Give a reason why we may or may not have a fundamental set of solutions.

A. $3x^2y'' + 6xy' + y = 0, \quad x > 0$

Known: two linearly independent functions on $x > 0$.

What can go wrong? *functions are not necessarily solutions*

B. $(x-2)y''' + x^2y' + 6y = 0, \quad x > 2$

Known: two linearly independent solutions on $x > 2$.

What can go wrong? *need three linearly independent solutions*

C. $x^2y'' + 5xy' + 6y = 0, \quad x > 0$

Known: two solutions on $x > 0$.

What can go wrong? *Solutions are not necessarily linearly independent*

D. $x^4y^{(4)} + 6x^3y''' + 9x^2y'' + 3xy' + y = 0, \quad x > 0$

Known: four linearly independent solutions on $(0, \infty)$.

What can go wrong? *Nothing*

E. $2x^3y''' + xy' + y = 0, \quad x > 0$

Known: three linearly independent solutions on $(-\infty, \infty)$.

What can go wrong? *One solution is the absolute value of another solution*

2. Use reduction of order to find a second linearly independent solution to the differential equation $(x-1)y''-xy'+y=0$, $x>1$, where $y_1(x)=e^x$ is a known solution. (**NO POINTS** for the integral formula).

$$y_2(x) = u(x)e^x$$
$$y_2'(x) = u'(x)e^x + u(x)e^x$$
$$y_2''(x) = u''(x)e^x + 2u'(x)e^x + u(x)e^x$$
$$\Rightarrow (x-1)(u''e^x + 2u'e^x + ue^x) - x(u'e^x + ue^x) + ue^x = 0$$

Dividing by e^x and regrouping by u,

$$(x-1)u'' + \underbrace{(2(x-1)-x)}_{(x-2)}u' + \underbrace{((x-1)-x+1)}_{0}u = 0$$

Let $\begin{cases} w = u' \\ w' = u'' \end{cases}$

$$(x-1)\frac{dw}{dx} + (x-2)w = 0$$

$$\Rightarrow \frac{dw}{w} = -\frac{(x-2)}{x-1}dx$$

$$\int \frac{dw}{w} = -\int \frac{(x-2)}{x-1}dx = -\int \frac{(x-1-1)}{x-1}dx = \int \left(-1 + \frac{1}{x-1}\right)dx$$

$$\ln|w| = -x + \ln|x-1| + C$$

$$w = e^{-x + \ln|x-1| + C} = C_1(x-1)e^{-x}$$

$$u(x) = C_1 \int (x-1)e^{-x}dx \qquad\qquad s = (x-1) \qquad dt = e^{-x}dx$$
$$\qquad\qquad\qquad\qquad\qquad ds = dx \qquad\quad t = -e^{-x}$$
$$= C_1\left[-(x-1)e^{-x} + \int e^{-x}dx\right]$$
$$= C_1\left[-xe^{-x} + e^{-x} - e^{-x}\right] + C_2$$
$$= \tilde{C}_1 xe^{-x} + C_2 \qquad\qquad \text{Let } \begin{cases} \tilde{C}_1 = 1 \\ C_2 = 0 \end{cases}$$
$$= xe^{-x}$$

So $y_2(x) = u(x) y_1(x)$
$$= xe^{-x}e^x$$
$$= x$$

3. Solve the differential equation $y^{(4)} - y''' - y' + y = 0$.

$$\text{Assume } y(x) = e^{rx}$$
$$y'(x) = re^{rx}$$
$$y''(x) = r^2 e^{rx}$$
$$y'''(x) = r^3 e^{rx}$$
$$y^{(iv)}(x) = r^4 e^{rx}$$

$$\Rightarrow e^{rx}[r^4 - r^3 - r + 1] = 0$$
$$r^4 - r^3 - r + 1 = 0$$
$$r^3(r-1) - (r-1) = 0$$
$$(r-1)(r^3 - 1) = 0$$
$$(r-1)(r-1)(r^2 + r + 1) = 0$$

$$\Rightarrow r = 1, 1 \qquad r = \frac{-1 \pm \sqrt{1 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$y(x) = c_1 e^x + c_2 x e^x + e^{-\frac{1}{2}x}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

4. Find a particular solution y_p to the differential equation $y'' - 9y = 4e^{3x} + 3x$ using the method of undetermined coefficients.

$$y'' - 9y = 0 \quad \text{Assume } y(x) = e^{rx}$$
$$\Rightarrow e^{rx}(r^2 - 9) = 0$$
$$(r+3)(r-3) = 0$$
$$r = 3, -3$$
$$\Rightarrow y_h = c_1 e^{3x} + c_2 e^{-3x}$$

$$g(x) = 4e^{3x} + 3x$$
$$r = 3, 0, 0$$
$$\Rightarrow r^2(r-3) = 0$$
$$\Rightarrow D^2(D-3)[g(x)] = 0$$

$$D^2(D-3)(D^2-9)y = D^2(D-3)[g(x)]$$
$$= 0$$

Assume $y(x) = e^{rx}$
$$\Rightarrow r^2(r-3)(r+3)(r-3) = 0$$
$$r = 3, -3, 3, 0, 0$$
$$y(x) = \underbrace{c_1 e^{3x} + c_2 e^{-3x}}_{y_h} + \underbrace{c_3 x e^{3x} + c_4 + c_5 x}_{y_p}$$

$$\Rightarrow y_p = Axe^{3x} + B + Cx$$
$$y_p' = Ae^{3x} + 3Axe^{3x} + C$$
$$y_p'' = 6Ae^{3x} + 9Axe^{3x}$$

$$y_p'' - 9y_p = (9A - 9A)xe^{3x} + 6Ae^{3x} - 9B - 9Cx$$
$$= 6Ae^{3x} - 9B - 9Cx$$
$$= 4e^{3x} + 3x$$
$$6A = 4 \Rightarrow A = \frac{2}{3}$$
$$-9B = 0 \Rightarrow B = 0$$
$$-9C = 3 \Rightarrow C = -\frac{1}{3}$$

$$\Rightarrow y_p = \frac{2}{3}xe^{3x} - \frac{x}{3}$$

5. A model for the population P(t) in a suburb of a large city is given by the initial value problem

$$\frac{dP}{dt} = P(10^{-1} - 10^{-7}P), \quad P(0) = 5000,$$

where t is measured in months.

A. What is the limiting value of the population?

$$\frac{dP}{dt} = 10^{-7}P(10^6 - P) = 0$$

$$P = 0, 10^6$$

Interval	TV	+/-	↑/↓
$(-\infty, 0)$	-10^6	$-$	↓
$(0, 10^6)$	10^5	$+$	↑
$(10^6, \infty)$	10^7	$-$	↓

$P = 10^6$ asymptotically stable (limiting value)

$P = 0$ unstable

B. At what time will the population be equal to one-half of this limiting value? (Hint: use the implicit solution)

$$\frac{dP}{dt} = 10^{-7}P(10^6 - P)$$

$$\frac{10^7 dP}{P(10^6 - P)} = dt$$

$$\frac{10^7}{P(10^6 - P)} = \frac{A}{P} + \frac{B}{10^6 - P}$$

$$10^7 = A(10^6 - P) + BP$$

$$P = 0 \quad 10^7 = 10^6 A \Rightarrow A = 10$$

$$P = 10^6 \quad 10^7 = 10^6 B \Rightarrow B = 10$$

$$10\int\left(\frac{1}{P} + \frac{1}{10^6 - P}\right)dP = \int dt$$

$$10[\ln|P| - \ln|10^6 - P|] = t + c$$

$$10\ln\left|\frac{P}{10^6 - P}\right| = t + c$$

$$10\ln\left|\frac{5000}{10^6 - 5000}\right| = 0 + c$$

$$\Rightarrow c = 10\ln\left|\frac{1}{199}\right|$$

Let $P(t_1) = \frac{10^6}{2}$

$$10\ln\left|\frac{\frac{10^6}{2}}{10^6 - \frac{10^6}{2}}\right| = t_1 + 10\ln\left|\frac{1}{199}\right|$$

$$10\ln|1| = 0 = t_1 + 10\ln\left|\frac{1}{199}\right|$$

$$\Rightarrow t_1 = -10\ln\left|\frac{1}{199}\right|$$

$$\approx 52.9 \text{ months}$$

Bonus: (10 points) Two chemicals A and B are combined to form a chemical C. The rate of the reaction is proportional to the product of the instantaneous amounts of A and B not converted to C. Initially, there are 40 grams of A and B each, and for every 2 grams of B, 1 gram of A is used. It is observed after 10 minutes that 5 grams of C are formed. Set up, **but don't solve** the differential equation with the appropriate "boundary" conditions.

Let $X(t)$ = amount of C at any time t (minutes).

$$\text{amount of A left} = 40 - \frac{1}{1+2}x = 40 - \frac{x}{3}$$

$$\text{amount of B left} = 40 - \frac{2}{1+2}x = 40 - \frac{2x}{3}$$

$$\Rightarrow \frac{dX}{dt} = K\left(40 - \frac{x}{3}\right)\left(40 - \frac{2x}{3}\right) = K_1(120-x)(60-x) \qquad K_1 = \frac{2K}{9}$$

$$\begin{cases} X(0) = 0 \\ X(10) = 5. \end{cases}$$