MTH 204 Sections C & F
Spring 2008
Exam 2

(Wintz)

Name: Key

Section: C or F (circle one)

Read the directions carefully.

Write neatly in pencil and show all your work (you will only get credit for what you put on paper).

Please do not share calculators during the test.

Each question is worth 20 points

DO NOT USE Decimals on any intermediate step.

The last page contains your Laplace tables.

If you have trouble during the test, feel free to ask me for help.

Score:_______

1. Consider the differential equation $x^2y'' - 4xy' = x^5$.

   a. Classify the differential equation by order, linearity, type of coefficients, and state whether or not the equation is homogeneous.

   2nd order, linear, variable coefficients, nonhomogeneous

   Cauchy-Euler

   b. What method(s) can you use to solve this equation?

   $\begin{cases} \text{VOP} \\ \text{MUC (not recommended)} \end{cases}$

   c. Solve the equation for the interval $x > 0$.

Method 1: VOP

1. Solve $x^2y'' - 4xy' = 0$ Let $y(x) = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

   $\Rightarrow \cancel{x^m}[m(m-1) - 4m] = m^2 - 5m = m(m-5) = 0 \Rightarrow m = 0, 5$

   $\Rightarrow y_h(x) = c_1 + c_2x^5$

2. $W(y_1, y_2) = \begin{vmatrix} 1 & x^5 \\ 0 & 5x^4 \end{vmatrix} = 5x^4$

3. Std form: $y'' - \frac{4}{x}y' = x^3 = f(x)$

4. Let $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$

"5." Plug in.

6. Cramer's Rule: $W_1 = \begin{vmatrix} 0 & x^5 \\ x^3 & - \end{vmatrix} = -x^8$, $W_2 = \begin{vmatrix} 1 & 0 \\ 0 & x^3 \end{vmatrix} = x^3$

   $\Rightarrow u_1'(x) = \frac{W_1}{W(y_1, y_2)} = -\frac{x^4}{5}$, $u_2'(x) = \frac{W_2}{W(y_1, y_2)} = \frac{1}{5x}$.

7. Integrate: $u_1(x) = -\frac{x^5}{25}$, $u_2(x) = \frac{1}{5}\ln x$

   $\Rightarrow y_p(x) = \underbrace{-\frac{x^5}{25}(1)}_{\text{absorbed into } y_1} + \left(\frac{1}{5}\ln x\right)x^5 = \frac{1}{5}x^5\ln x$.

8. GS: $y(x) = y_h(x) + y_p(x)$

   $= c_1 + c_2x^5 + \frac{1}{5}x^5\ln x$.

Method2: MUC

1. Solve $x^2y'' - 4xy' = 0$

$$y_h(x) = c_1 + c_2x^5$$

2. Find y_p

$$y_p(x) = Ax^5 \ln x$$

$$y_p' = 5Ax^4 \ln x + Ax^4$$

$$y_p'' = 20Ax^3 \ln x + 5Ax^3 + 4Ax^3$$

$$= 20Ax^3 \ln x + 9Ax^3$$

3. MUC

$$x^2y_p'' - 4xy_p' = x^2(20Ax^3 \ln x + 9Ax^3) - 4x(5Ax^4 \ln x + Ax^4)$$

$$= (20A - 20A)x^5 \ln x + (9A - 4A)x^5$$

$$= 5Ax^5$$

$$= x^5$$

$\Rightarrow A = 1/5$

$\Rightarrow y_p(x) = \frac{1}{5}x^5 \ln x$

4. GS

$$y(x) = y_h(x) + y_p(x)$$

$$= c_1 + c_2x^5 + \frac{1}{5}x^5 \ln x$$

2. Suppose a 24 lb object stretches a vertical spring 2 ft to equilibrium position when it is first attached. Initially the object is released 3 ft above equilibrium position with a downward velocity of 6 ft/s. Assume there is no damping force, but there is an external force of $f(t) = 12\sin(\gamma t)$.

a. Set up, **do not solve**, the IVP describing this motion. **Use** $g = 32\text{ft}/s^2$ as the acceleration for gravity.

$m = \frac{W}{g} = \frac{24\text{lb}}{32\text{ft/s}^2} = \frac{3}{4}\text{slug}$

$\beta = 0$

EP: $mg = ks \Rightarrow 24\text{lb} = 2k\text{ft}$

$\Rightarrow k = 12\text{lb/ft}$

$\begin{cases} y(0) = -3 \\ y'(0) = 6 \end{cases}$

$\frac{3}{4}y'' + 12y = 12\sin(\gamma t)$

$\begin{cases} y(0) = -3 \\ y'(0) = 6 \end{cases}$

b. For what value of γ does resonance occur?.

$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{12}{3/4}} = \sqrt{16} = 4 = \gamma$

c. Rewriting the above spring-mass problem as a circuit problem, give the inductance, resistance, capacitance, and impressed voltage. Make sure you carefully label each term.

Recall: $Lq'' + Rq' + \frac{1}{C}q = E(t)$

$L = \frac{3}{4}h$

$R = 0\Omega$

$C = \frac{1}{12}f$

$E(t) = 12\sin(\gamma t)$

3. Find a particular solution to the differential equation $y''' - y = e^x$. (**Hint:** you may find the identity $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ useful).

2nd order, linear, constant coefficients, nonhomogeneous

Methods: { MUC
VOP (not recommended, 3x3 Wronskian)
Laplace (not recommended, messy partial fractions)

1. Solve $y''' - y = 0$, $y(x) = e^{rx}$

$\Rightarrow \cancel{e^{rx}}[r^3 - 1] = (r-1)(r^2+r+1) = 0 \Rightarrow r = 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

$\Rightarrow y_h(x) = e^{-\frac{1}{2}x}\left[c_1\cos\left(\frac{\sqrt{3}}{2}x\right) + c_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right] + c_3e^x$

2. Root(s) of $g(x)$: $r = 1$
3. Characteristic Eq that gives $g(x)$: $r - 1 = 0$
4. Annihilator: $D - 1$
5. $(D-1)(D-1)(D^2+D+1)y = (D-1)e^x = 0$
6. $\cancel{e^{rx}}(r-1)(r-1)(r^2+r+1) = 0 \Rightarrow r = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i, 1, 1$

$\Rightarrow y(x) = \underbrace{e^{-\frac{1}{2}x}\left[c_1\cos\left(\frac{\sqrt{3}}{2}x\right) + c_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right] + c_3e^x}_{y_h} + \underbrace{c_4xe^x}_{y_p}$

7. $y_p = Axe^x$

$y_p' = Ae^x + Axe^x$

$y_p'' = 2Ae^x + Axe^x$

$y_p''' = 3Ae^x + Axe^x$

$y_p''' - y_p = (A - A)xe^x + 3Ae^x = 3Ae^x = e^x$

$\Rightarrow A = 1/3$

$\Rightarrow y_p = \frac{1}{3}xe^x$

4. Find the inverse Laplace transform of the following:

a. $F(s) = \left\{ \frac{(s+1)^2}{s^4} \right\}$

$$\mathcal{L}^{-1}\left\{\frac{(s+1)^2}{s^4}\right\} = \mathcal{L}^{-1}\left\{\frac{s^2+2s+1}{s^4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s^{2+1}}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^{3+1}}\left(\frac{3!}{3!}\right)\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{2!}{s^{2+1}}\right\} + \frac{1}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{s^{3+1}}\right\}$$

$$= t + t^2 + \frac{1}{6}t^3$$

b. $G(s) = \frac{s}{s^2 + 4s + 10}$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4s+10}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4s+4-4+10}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+2-2}{(s+2)^2+6}\right\} = \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+6}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+6}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+(\sqrt{6})^2}\bigg|_{s \to s+2}\right\} - \frac{2}{\sqrt{6}}\mathcal{L}^{-1}\left\{\frac{\sqrt{6}}{s^2+(\sqrt{6})^2}\bigg|_{s \to s+2}\right\}$$

$$= e^{-2t}\cos(\sqrt{6}t) - \sqrt{\frac{2}{3}}\,e^{-2t}\sin(\sqrt{6}t).$$

5. Solve the IVP $y'+y = \mathcal{U}(t-1)$ subject to $y(0) = 2$. Graph your solution.

1st order, linear, constant coefficients, nonhomogeneous

$$\text{Methods} \begin{cases} \text{VOP (see Ex6 in Sec 2.3)} \\ \text{Laplace (recommended)} \end{cases}$$

1. Take Laplace of both sides

$$\mathcal{L}\{y'+y\} = \mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{\mathcal{U}(t-1)\}$$

$$sY(s) - \overset{=2}{y(0)} + Y(s) = \frac{e^{-s}}{s}$$

2. Solve for $Y(s)$

$$(s+1)Y(s) = \frac{e^{-s}}{s} + 2 \Rightarrow Y(s) = \frac{e^{-s}}{s(s+1)} + \frac{2}{s+1}$$

Let $F(s) = \frac{1}{s(s+1)}$

$$= e^{-s}F(s) + \frac{2}{s+1}$$

3. Partial Fractions $F(s) = \frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{1}{s} - \frac{1}{s+1}$

$1 = A(s+1) + Bs$

$s=0 \Rightarrow A=1$

$s=-1 \Rightarrow B=-1$

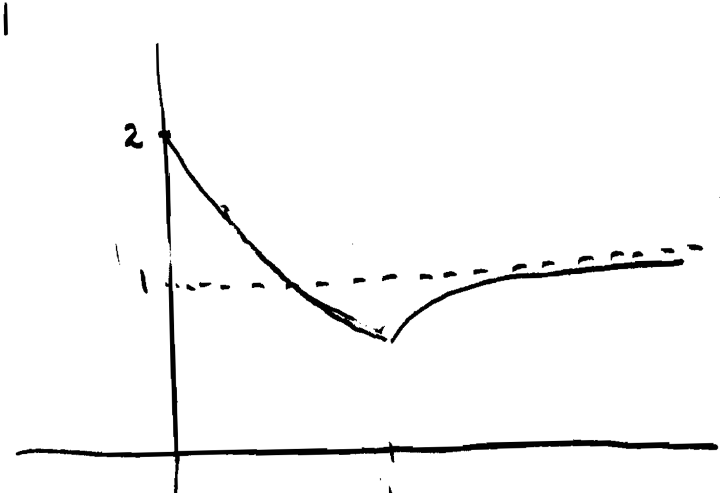
Note $f(t) = 1 - e^{-t}$

4. Inverse Laplace

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{e^{-s}F(s)\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= f(t-1)\mathcal{U}(t-1) + 2e^{-t} = (1 - e^{-(t-1)})\mathcal{U}(t-1) + 2e^{-t}$$

or $y(t) = \begin{cases} 2e^{-t}, & 0<t<1 \\ 2e^{-t} + 1 - e^{-(t-1)}, & t \geq 1 \end{cases}$



Bonus (10 points): Solve the differential equation $y^{(5)} + 5y^{(4)} - 2y''' - 10y'' + y' + 5y = 0$.

Assume $y(x) = e^{rx}$

$\Rightarrow \cancel{e^{rx}}\left[r^5 + 5r^4 - 2r^3 - 10r^2 + r + 5\right] = 0$

$$
\begin{array}{r|l}
 & r^4 - 2r^2 + 1 \\
\hline
r+5 & r^5 + 5r^4 - 2r^3 - 10r^2 + r + 5 \\
 & \underline{r^5 + 5r^4} \\
 & 0 - 2r^3 - 10r^2 \\
 & \underline{-2r^3 - 10r^2} \\
 & 0 + r + 5 \\
 & \underline{r + 5} \\
 & 0
\end{array}
$$

$\Rightarrow (r+5)(r^4 - 2r^2 + 1) = 0$

$(r+5)(r^2-1)^2 = (r+5)(r-1)^2(r+1)^2 = 0$

$\Rightarrow r = -1, -1, 1, 1, -5$

$y(x) = c_1 e^{-x} + c_2 x e^{-x} + c_3 e^{x} + c_4 x e^{x} + c_5 x e^{-5x}$