MTH 204
Spring 2009
Exam 2

Name: Key

Section: C or F (circle one)

Read the directions carefully.
Write neatly in pencil and show all your work
(you will only receive credit for what you put on your paper).
Please do not share calculators during the test.
Each question is worth 20 points.
DO NOT USE decimals on any intermediate step.
The last page contains your Laplace tables.
If you have trouble during the test, feel free to ask me for help.

Score:________

1. Consider the differential equation $y''' - y = 2e^t$.

a. Classify the differential equation by order, linearity, type of coefficients, and whether or not the equation is homogeneous or nonhomogeneous.

3rd order, linear, constant coefficients, nonhomogeneous

b. What method(s) can you use to solve this equation? What are the potential drawbacks to each method?

MUC - Algebra/ Product rule
VOP - 3x3 Wronskians
Laplace - messy Partial fractions / Need to fudge ICs

c. Find the general solution. You may find the identity $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ useful.

1. Solve $y''' - y = 0 \Rightarrow y(t) = e^{rt} \Rightarrow r^3 - 1 = 0$
$\Rightarrow (r-1)(r^2+r+1) = 0 \Rightarrow r = \frac{-1 \pm \sqrt{1-4}}{2(1)} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$
$y_h(t) = e^{-\frac{1}{2}t}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right] + c_3 e^t$
2. Roots of $g(t)$: $r = 1$
3. Char Eq: $r - 1 = 0$
4. Ann: $D - 1$
5. $(D-1)(D^3-1)y = (D-1)e^t = 0$ 4th, linear, CC, Hom
6. Solve $\Rightarrow y(t) = e^{rt} \Rightarrow (r-1)(r-1)(r^2+r+1) = 0$
$y(t) = \underbrace{e^{-\frac{1}{2}t}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right] + c_3 e^t}_{y_h} + \underbrace{c_4 t e^t}_{y_p}$
7. $y_p(t) = Ate^t$ $\quad y_p''(t) = 2Ae^t + Ate^t$
$y_p'(t) = Ae^t + Ate^t$ $\quad y_p'''(t) = 3Ae^t + Ate^t$
8. MUC: $y_p''' - y_p = 3Ae^t + (\cancel{A-A})te^t = 2e^t \Rightarrow A = \frac{2}{3}$
$\Rightarrow y_p(t) = \frac{2}{3}te^t$

9. GS
$y(t) = e^{-\frac{1}{2}t}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right] + c_3 e^t + \frac{2}{3}te^t$

2. A 16-lb object stretches a spring 2 ft in equilibrium position. Initially the object is released 1 ft below equilibrium position with an upward velocity of 3 ft/s, and the subsequent motion takes place in a medium that offers a damping force numerically 7 times the instantaneous velocity. Assume that the object is driven by the external force $f(t) = 10\cos(\gamma t)$.

a. Set up, but **<u>don't solve</u>** the IVP describing this motion. <u>Use</u> $g(t) = 32\ ft/s^2$ as the acceleration due to gravity.

$$m = \frac{W}{g} = \frac{16}{32} = \frac{1}{2} \text{ slugs}$$

$$\beta = 7$$

$$mg = Ks \Rightarrow 16 = 2K \Rightarrow K = 8 \text{ lb/ft}$$

$$\begin{cases} \frac{1}{2}y'' + 7y' + 8y = 10\cos(\gamma t) \\ y(0) = 1 \\ y'(0) = -3 \end{cases}$$

b. Rewriting the above spring-mass problem as a circuit problem, give the inductance, resistance, capacitance, and the impressed voltage. Make sure that you carefully label each term.

$$L = \frac{1}{2} \text{ h}$$

$$R = 7\,\Omega$$

$$C = \frac{1}{8} \text{ f}$$

$$E(t) = 10\cos(\gamma t) \text{ V}$$

c. Now remove the damping force. For what value γ does resonance occur?

$$\omega = \sqrt{\frac{K}{m}} = \sqrt{\frac{8}{1/2}} = 4$$

$$\omega = \gamma = 4$$

3. Consider the differential equation $x^2y'' + xy' - y = 3x, \quad x > 0.$

a. Classify the differential equation by order, linearity, type of coefficients, and whether or not the equation is homogeneous or nonhomogeneous.

2nd order, linear, variable coefficients, nonhomogeneous (Cauchy-Euler)

b. What method(s) can you use to solve this equation?

$\begin{cases} \text{MUC - not recommended} \\ \text{VOP} \end{cases}$

c. Find a particular solution.

1. Solve $x^2y''+xy'+x=0$, $y(x)=x^m$

$\Rightarrow \cancel{x^m}[m^2+(1-1)m-1] = \cancel{x^m}(m^2-1)=0 \quad \Rightarrow m=\pm 1$

$\Rightarrow y_h(x) = c_1 x + c_2 x^{-1}$

2. $W(y_1,y_2) = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} = -x^{-1} - x^{-1} = -2x^{-1}$

3. Std form: $y'' + x^{-1}y' - x^{-2}y = 3x^{-1} = f(x)$

4. Assume $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$

"5" Plug in y_p

6. Cramer's Rule

$W_1 = \begin{vmatrix} 0 & x^{-1} \\ 3x^{-1} & - \end{vmatrix} = -3x^{-2} \Rightarrow u_1'(x) = \frac{W_1}{W(y_1,y_2)} = \frac{-3x^{-2}}{-2x^{-1}} = \frac{3}{2}x^{-1}$

$W_2 = \begin{vmatrix} x & 0 \\ - & 3x^{-1} \end{vmatrix} = 3 \Rightarrow u_2'(x) = \frac{W_2}{W(y_1,y_2)} = \frac{3}{-2x^{-1}} = -\frac{3}{2}x$

7. Integrate & plug in

$u_1(x) = \frac{3}{2}\int \frac{dx}{x} = \frac{3}{2}\ln x + \cancel{K_1}$

$u_2(x) = -\frac{3}{2}\int x\,dx = -\frac{3}{4}x^2 + \cancel{K_2}$

absorbed by $y_1(x)$

$\Rightarrow y_p(x) = \left(\frac{3}{2}\ln x\right)x - \left(\frac{3}{4}x^2\right)x^{-1} = \frac{3}{2}x\ln x - \frac{3}{4}x$

$= \frac{3}{2}x\ln x$

4. Find the inverse Laplace of the following.

a. $F(s) = \dfrac{(s-1)^2}{s^4}$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s^2-2s+1}{s^4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{2!}{s^{2+1}}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^{3+1}}\left(\frac{3!}{3!}\right)\right\}$$

$$= t - t^2 + \frac{1}{6}t^3$$

b. $G(s) = \dfrac{s}{s^2+2s+17}$

$$\mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+1-1+17}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2+4^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+1-1}{(s+1)^2+4^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}_{s \to s-(-1)} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{4}{s^2+4^2}\right\}_{s \to s-(-1)}$$

$$= e^{-t}\cos(4t) - \frac{1}{4}e^{-t}\sin(4t)$$

5. Consider the IVP $y' + 2y = \mathscr{U}(t-3)$, $y(0) = 1$.

a. Solve the IVP.

1. Take the Laplace of both sides

$$\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\mathcal{U}(t-3)\}$$

$$(sY(s) - \underset{\overset{\parallel}{1}}{y(0)}) + 2Y(s) = \frac{e^{-3s}}{s}$$

$$(s+2)Y(s) - 1 = \frac{e^{-3s}}{s}$$

2. Solve for $Y(s)$

$$Y(s) = \frac{1}{s+2} + \frac{e^{-3s}}{s(s+2)} = \frac{1}{s+2} + e^{-3s}F(s) \text{ where } F(s) = \frac{1}{s(s+2)}$$

3. Find $f(t)$

$$\frac{1}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} = \frac{\frac{1}{2}}{s} - \frac{\frac{1}{2}}{s+2} = F(s)$$

$$\Rightarrow 1 = A(s+2) + Bs$$

$$s = 0 \Rightarrow A = \tfrac{1}{2}$$

$$s = -2 \Rightarrow B = -\tfrac{1}{2}$$

$$\Rightarrow f(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$$

4. Find $y(t)$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \mathcal{L}^{-1}\{e^{-3s}F(s)\}$$

$$= e^{-2t} + f(t-3)\,\mathcal{U}(t-3) = e^{-2t} + \frac{1}{2}(1 - e^{-2(t-3)})\mathcal{U}(t-3)$$

$$= \begin{cases} e^{-2t}, & 0 \le t < 3 \\ e^{-2t} + \frac{1}{2} - \frac{1}{2}e^{-2(t-3)}, & t \ge 3 \end{cases}$$

b. Evaluate the following.

$y(1) =$ e^{-2}

$y(4) =$ $e^{-8} + \frac{1}{2} - \frac{1}{2}e^{-2}$

Bonus (10 points): Find the smallest annihilator of the function

$$g(x) = 3x^2 - xe^{-4x} + 6\cos(3x) + 12e^x \sin(3x).$$

Roots: $r = 0, 0, 0, -4, -4, 0 \pm 3i, 1 \pm 3i$

Char Eq: $r^3 (r+4)^2 (r^2+9)[(r-1)^2+9] = 0$

Ann: $D^3 (D+4)^2 (D^2+9)[(D-1)^2+9]$