MTH 204
Spring 2006
Exam 2
sections D & E

Name: Key

Section: D\E

Read the directions carefully.
Each question is worth 20 points.
Write neatly in pencil and show all your work
(you will only get credit for what you put on paper).
Do not use decimals in any intermediate step.
Please do not share calculators during the test.
If you have trouble during the test, feel free to ask me for help.

1. Consider the differential equation $2x^2y''+5xy'+y=0$ on the interval $(0,\infty)$.

A. Give the three conditions needed to have a fundamental set of solutions to this equation on $(0,\infty)$.

i. y_1, y_2 are solutions to the DE
ii. y_1, y_2 are linearly independent
iii. # linearly independent solutions = order.

B. Consider the functions $f(x)=x^{-1/2}$, $g(x)=3x^2$, and $h(x)=2x^{-1}$. Do f(x) and g(x) form a fundamental set of solutions? Why or why not? Do f(x) and h(x) form a fundamental set of solutions? Why or why not?

$f(x)=x^{-1/2}$ $\quad g(x)=3x^2$ $\quad h(x)=2x^{-1}$

$f'(x)=-\frac{1}{2}x^{-3/2}$ $\quad g'(x)=6x$ $\quad h'(x)=-2x^{-2}$

$f''(x)=\frac{3}{4}x^{-5/2}$ $\quad g''(x)=6$ $\quad h''(x)=4x^{-3}$

then $2x^2f''+5xf'+f=2x^2\left(\frac{3}{4}x^{-5/2}\right)+5x\left(-\frac{1}{2}x^{-3/2}\right)+x^{-1/2}=\frac{3}{2}x^{-1/2}-\frac{5}{2}x^{-1/2}+x^{-1/2}=0 \Rightarrow f$ is a solution

$2x^2g''+5xg'+g=2x^2(6)+5x(6x)+3x^2=12x^2+30x^2+3x^2=45x^2\neq 0 \Rightarrow g$ is not a solution

$2x^2h''+5xh'+h=2x^2(4x^{-3})+5x(-2x^{-2})+2x^{-1}=8x^{-1}-10x^{-1}+2x^{-1}=0 \Rightarrow h$ is a solution

$f(x), g(x)$ do not form a FSS since $g(x)$ is not a solution.

$$W(f(x),h(x))=\begin{vmatrix} f & h \\ f' & h' \end{vmatrix}=\begin{vmatrix} x^{-1/2} & 2x^{-1} \\ -\frac{1}{2}x^{-1/2} & -2x^{-2} \end{vmatrix}=x^{-1/2}(-2x^{-2})-\left(-\frac{1}{2}x^{-1/2}\right)(2x^{-1})$$

$$=-2x^{-3/2}+x^{-3/2}=-x^{-3/2}\neq 0 \Rightarrow f, h \text{ are linearly independent.}$$

Then since $f(x), h(x)$ are linearly independent solutions to a 2nd order equation, they form a FSS.

2. Newton's law of cooling states that the rate of change in the temperature of a body is proportional to the difference in the temperature of that body and temperature of the surrounding medium. Now suppose that a murder victim is discovered at midnight with a recorded temperature of 31° C. An hour later, the temperature of the victim is 29° C. Assume that the temperature of the surrounding air remains a constant 21° C. Calculate the victim's time of death. <u>Note</u>: the "normal" temperature of a living person is 37° C.

$$\frac{dT}{dt} = K(T-21) \qquad \begin{cases} T(0) = 31 \\ T(1) = 29 \end{cases}$$

$$\int \frac{dT}{T-21} = \int k\,dt$$

$$\ln|T-21| = Kt + c$$

$$T - 21 = e^{Kt+c} = c_1 e^{Kt}$$

$$T(t) = 21 + c_1 e^{Kt}$$

$$T(0) = 31 = 21 + c_1 e^0 \Rightarrow c_1 = 10$$

$$\text{So } T(t) = 21 + 10e^{Kt}$$

$$T(1) = 29 = 21 + 10e^K$$

$$8 = 10e^k$$

$$\frac{4}{5} = e^k \Rightarrow K = \ln\left(\frac{4}{5}\right)$$

$$\text{So } T(t) = 21 + 10e^{\ln\left(\frac{4}{5}\right)t}$$

Let t_0 = time of death.

$$T(t_0) = 37 = 21 + 10e^{\ln(4/5)t_0}$$

$$16 = 10e^{\ln(4/5)t_0}$$

$$\frac{8}{5} = e^{\ln(4/5)t_0}$$

$$\ln\left(\frac{8}{5}\right) = \ln\left(\frac{4}{5}\right)t_0$$

$$\Rightarrow t_0 = \frac{\ln(8/5)}{\ln(4/5)} \approx -2.10628$$

$$\left(-2.10628\text{hrs}\right)\left(\frac{60\text{min}}{1\text{hr}}\right)$$

$$= -126.377 \text{ min.}$$

Time of death: 9:54 PM

3. $y_1(x) = x^4$ is a solution to the differential equation $x^2y''-7xy'+16y=0$. Use reduction of order to find a second linearly independent solution on the interval $(0,\infty)$. Give the general solution. <u>No points</u> will be awarded if you use the integral formula.

$y_2(x) = u(x)\,y_1(x) = ux^4$

$y_2' = u'x^4 + 4ux^3$

$y_2'' = u''x^4 + 4u'x^3 + 4u'x^3 + 12ux^2 = u''x^4 + 8u'x^3 + 12ux^2$

Plugging y_2 into the DE we get

$x^2[u''x^4+8u'x^3+12ux^2] - 7x[u'x^4+4ux^3] + 16ux^4 = 0$

Regrouping in terms of u,

$u''x^6 + u'\underbrace{[8x^5 - 7x^5]}_{x^5} + u\underbrace{[12x^4 - 28x^4 + 16x^4]}_{0} = 0$

$x^6u'' + x^5u' = 0$

Let $w = u'$

$w' = u''$

$x^6\dfrac{dw}{dx} + x^5w = 0$ <u>Note</u>: this is a 1st order, linear, <u>separable</u> equation

$x^6\dfrac{dw}{dx} = -x^5w$

$\int \dfrac{dw}{w} = -\int \dfrac{1}{x}dx.$

$\ln|w| = -\ln|x| = \ln|x^{-1}|$

$\Rightarrow w = x^{-1}$

But $w = u'$

So $u(x) = \int x^{-1}dx = \ln x$

Now $y_2(x) = u(x)\,y_1(x) = x^4\ln x$

So the general solution is $y(x) = c_1x^4 + c_2x^4\ln x$

4. Two chemicals A and B are combined to form a chemical C. The rate of the reaction is proportional to the product of the instantaneous amounts of A and B not converted to C. Initially, there are 30 grams of A and B each, and for every 2 grams of B, 3 grams of A is used. It is observed after 10 minutes that 5 grams of C are formed.

A. Set up the differential equation with the appropriate "boundary" conditions.

Let $X(t)$ = amount of C at time t.

$$\frac{dx}{dt} = K\left(a - \frac{M}{M+N}x\right)\left(b - \frac{N}{M+N}x\right) \qquad \begin{cases} X(0) = 0 \\ X(10) = 5 \end{cases}$$

$$= K\left(30 - \frac{3}{2+3}x\right)\left(30 - \frac{2}{2+3}x\right)$$

$$= K\left(30 - \frac{3}{5}x\right)\left(30 - \frac{2}{5}x\right)$$

$$= K_1(50 - x)(75 - x) \quad \text{where } K_1 = \frac{6}{25}K, \quad K > 0$$

B. Give the order of the differential equation and state whether the equation is linear or nonlinear, separable or non-separable, and autonomous or non-autonomous.

1st order, nonlinear, separable, <u>autonomous</u>.

C. What is the limiting amount of C after a long time? How much of A and B are left over a long time? (Hint: What are the critical points?; **DO NOT** solve the DE)

$$\frac{dx}{dt} = K(50 - x)(75 - x) = 0 \qquad \text{Critical points: } x = 50, 75$$

Int	TV	+/-	↑/↓
$(-\infty, 50)$	0	+	↑
$(50, 75)$	60	−	↓
$(75, \infty)$	100	+	↑

$x = 75$ unstable

$x = 50$ asymptotically stable.

Limiting amount of C = 50g.

Remaining A = $30 - \frac{3}{5}(50) = 0$ g

Remaining B = $30 - \frac{2}{5}(50) = 10$g.

5. Find the general solution of $y^{(4)} + y''' + y'' = 0$.

Assume $y(x) = e^{rx}$

$y'(x) = re^{rx}$

$y''(x) = r^2 e^{rx}$

$y'''(x) = r^3 e^{rx}$

$y^{(4)}(x) = r^4 e^{rx}$

$r^4 e^{rx} + r^3 e^{rx} + r^2 e^{rx} = 0$

$e^{rx}(r^4 + r^3 + r^2) = 0$

$r^2(r^2 + r + 1) = 0$ since e^{rx} is never 0.

$r^2 = 0$

$r = 0, 0$

$r^2 + r + 1 = 0$

$$r = \frac{-1 \pm \sqrt{1 - 4(1)(1)}}{2(1)} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$y_1(x) = e^{0x} = 1$

$y_2(x) = xe^{0x} = x$

$y_3(x) = e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right)$

$y_4(x) = e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right)$

$$y(x) = c_1 + c_2 x + c_3 e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_4 e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right)$$

Bonus (10 points): $y_1(x) = e^x$ is a solution to the homogeneous differential equation

$y'' - 3y' + 2y = 0$. Use reduction of order to find a particular solution $y_p(x)$ that satisfies the

nonhomogeneous equation $y'' - 3y' + 2y = 5e^{3x}$.

$$y_p(x) = u(x)\,y_1(x) = ue^x$$

$$y_p' = u'e^x + ue^x$$

$$y_p'' = u''e^x + u'e^x + u'e^x + ue^x = u''e^x + 2u'e^x + ue^x$$

Plugging y_p into the nonhomogeneous equation

$$u''e^x + 2u'e^x + ue^x - 3[u'e^x + ue^x] + 2ue^x = 5e^{3x}$$

Regrouping in terms of u,

$$u''e^x + u'\underbrace{[2e^x - 3e^x]}_{-e^x} + u\underbrace{[e^x - 3e^x + 2e^x]}_{0} = 5e^{3x}$$

$$e^x u'' - e^x u' = 5e^{3x}$$

$$u'' - u' = \frac{5e^{3x}}{e^x} = 5e^{2x}$$

Let $w = u'$
$w' = u''$

$$\frac{dw}{dx} - w = 5e^{2x}$$

This is a 1st order, linear, <u>nonseparable</u>, nonhomogeneous equation.

$$P(x) = -1$$

$$IF = e^{-\int dx} = e^{-x}$$

$$e^{-x}\left[\frac{dw}{dx} - w = 5e^{2x}\right].$$

$$\Rightarrow \frac{d}{dx}\left[e^{-x}w\right] = 5e^x$$

$$e^{-x}w = \int 5e^x dx = 5e^x$$

$$\Rightarrow w = 5e^{2x}$$

$$u = \int 5e^{2x}dx = \frac{5}{2}e^{2x}$$

So $y_p(x) = u(x)\,y_1(x) = \left(\frac{5}{2}e^{2x}\right)(e^x) = \frac{5}{2}e^{3x}$