

MTH 204

Fall 2005

Exam 3

Name: Key

Section: A/C

Read the directions carefully.

Write neatly in pencil and show all your work
(you will only get credit for what you put on paper).

Please do not share calculators during the test.

DO NOT USE Decimals in any intermediate step.

If you have trouble during the test, feel free to ask me for help.

Score: _____

1. Consider the differential equation $x^2 y'' - xy' + y = 2x$

(20 points)

- a. Classify the differential equation by order, linearity, type of coefficients, and state whether or not the equation is homogeneous.

2nd order, linear, variable coefficients, nonhomogeneous

- b. What method(s) can you use to solve this equation?

y_h : Assume $y(x) = x^r$.

y_p : Variation of Parameters.

- c. Solve the equation for the interval $x > 0$.

$$x^2 y'' - xy' + y = 0$$

$$y(x) = x^r$$

$$y'(x) = r x^{r-1}$$

$$y''(x) = r(r-1) x^{r-2}$$

$$x^2(r(r-1)x^{r-2}) - x(rx^{r-1}) + x^r = 0$$

$$y_1(x) = x$$

$$y_2(x) = x \ln x$$

$$W(y_1, y_2) = \begin{vmatrix} x & x \ln x \\ 1 & \ln x + 1 \end{vmatrix} = x \ln x + x - x \ln x = x$$

Rewrite DE in standard form $y'' - \frac{y'}{x} + \frac{y}{x^2} = \frac{2}{x} = f(x)$

$$W_1 = \begin{vmatrix} 0 & x \ln x \\ 2x^{-1} & \ln x + 1 \end{vmatrix} = -2 \ln x$$

$$W_2 = \begin{vmatrix} x & 0 \\ 1 & 2x^{-1} \end{vmatrix} = 2$$

$$u_1'(x) = \frac{W_1}{W(y_1, y_2)} = \frac{-2 \ln x}{x}$$

$$u_2'(x) = \frac{W_2}{W(y_1, y_2)} = \frac{2}{x}$$

$$u_1(x) = -2 \int \frac{\ln x}{x} dx = -\frac{2v^2}{2} = -(\ln x)^2$$

$$u_2(x) = 2 \int \frac{dx}{x} = 2 \ln x$$

$$v = \ln x \quad dv = \frac{1}{x}$$

$$y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x) = -x(\ln x)^2 + 2x(\ln x)^2 = x(\ln x)^2$$

$$y(x) = y_h(x) + y_p(x) = c_1 x + c_2 x \ln x + x(\ln x)^2$$

2. An object weighing 4 lbs stretches a spring 1.5 in. The object is displaced 2 in. below the equilibrium position and is released with no initial velocity. Assuming that there is no damping force and that the object is acted on by an external force of $2\cos(3t)$ lb. Set up but **do not solve** the initial value problem describing this motion. Use $g = 32 \text{ ft/s}^2$ as the acceleration for gravity.

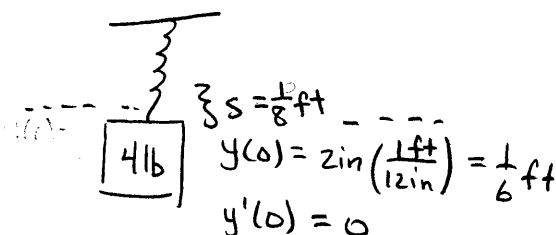
$$m = \frac{4}{32} = \frac{1}{8}$$

$$s = \frac{3}{2} \text{ in } \left(\frac{1 \text{ ft}}{12 \text{ in}} \right) = \frac{1}{8}$$

$$ks = mg \Rightarrow \frac{1}{8}k = 4 \\ \Rightarrow k = 32$$

$$my'' + ky = f(t)$$

$$\frac{1}{8}y'' + 32y = 2\cos(3t)$$



So the IVP is

$$\frac{1}{8}y'' + 32y = 2\cos(3t)$$

$$\begin{cases} y(0) = \frac{1}{6} \\ y'(0) = 0 \end{cases}$$

3. Use the definition of the Laplace transform ($\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt$) to find $\mathcal{L}\{f(t)\}$ for

those values of s for the improper integral converges when $f(t) = e^{-2t-5}$. (20 points)

$$\mathcal{L}\{e^{-2t-5}\} = \int_0^\infty e^{-st}e^{-2t-5}dt$$

$$= e^{-5} \int_0^\infty e^{-(s+2)t} dt$$

$$= e^{-5} \left(\frac{-e^{-(s+2)t}}{s+2} \right) \Big|_0^\infty$$

$$= e^{-5} \left(0 - \left(-\frac{1}{s+2} \right) \right)$$

$$= \frac{e^{-5}}{s+2}$$

provided $s+2 > 0$
 $s > -2$

4. Consider the differential equation $y'' + y = 4 \cos(x)$

(20 points)

a. Classify the differential equation by order, linearity, type of coefficients, and state whether or not the equation is homogeneous.

2nd order, linear, constant coefficients, nonhomogeneous

b. What method(s) can you use to solve this equation?

$y_h(x)$: Assume $y(x) = e^{rx}$

$y_p(x)$: MUC or VOP.

c. Find a particular solution, y_p .

$$y'' + y = 0$$

$$y(x) = e^{rx}$$

$$y'(x) = re^{rx}$$

$$y''(x) = r^2 e^{rx}$$

$$y_1(x) = \cos(x)$$

$$y_2(x) = \sin(x)$$

$$r^2 e^{rx} + e^{rx} = 0$$

$$r^2 + 1 = 0$$

$$r = \pm i$$

Method 1. MUC.

$$(D^2 + 1)(4 \cos(x)) = 0$$

$$(D^2 + 1)y = 4 \cos(x)$$

$$(D^2 + 1)(D^2 + 1)y = (D^2 + 1)[4 \cos(x)] = 0$$

Assume $y(x) = e^{rx}$

$$\Rightarrow (r^2 + 1)(r^2 + 1) = 0$$

$$r = \pm i, \pm i.$$

$$y(x) = \underbrace{c_1 \cos(x) + c_2 \sin(x)}_{y_h(x)} + \underbrace{c_3 x \cos(x) + c_4 x \sin(x)}_{y_p(x)}$$

$$y_p(x) = Ax \cos(x) + Bx \sin(x)$$

$$y_p'(x) = A \cos(x) - Ax \sin(x) + B \sin(x) + Bx \cos(x)$$

$$y_p''(x) = -A \sin(x) - A \sin(x) - Ax \cos(x) + B \cos(x) + B \cos(x) - Bx \sin(x) \\ = -2A \sin(x) - Ax \cos(x) + 2B \cos(x) - Bx \sin(x).$$

$$y_p'' + y_p = -2A \sin(x) - \cancel{Ax \cos(x)} + 2B \cos(x) - \cancel{Bx \sin(x)} + \cancel{Ax \cos(x)} + \cancel{Bx \sin(x)} \\ = 4 \cos(x)$$

$$-2A = 0 \Rightarrow A = 0$$

$$2B = 4 \Rightarrow B = 2$$

$$\text{So } y_p(x) = 2x \sin(x).$$

Method 2: VOP.

$$W(y_1, y_2) = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = \cos^2(x) + \sin^2(x) = 1.$$

$$W_1 = \begin{vmatrix} 0 & \sin(x) \\ 4\cos(x) & \cos(x) \end{vmatrix} = 4\sin(x)\cos(x) = 2\sin(2x)$$

$$W_2 = \begin{vmatrix} \cos(x) & 0 \\ -\sin(x) & 4\cos(x) \end{vmatrix} = 4\cos^2(x) = 4\left(\frac{1+\cos(2x)}{2}\right) = 2 + 2\cos(2x)$$

$$u_1'(x) = \frac{W_1}{W(y_1, y_2)} = \frac{-2\sin(2x)}{1}$$

$$u_2'(x) = \frac{W_2}{W(y_1, y_2)} = \frac{2 + 2\cos(2x)}{1}$$

$$u_1(x) = -2 \int \sin(2x) dx = -2 \left(-\frac{\cos(2x)}{2} \right) = \cos(2x)$$

$$u_2(x) = \int (2 + 2\cos(2x)) dx = \int 2 dx + 2 \int \cos(2x) dx = 2x + 2 \left(\frac{\sin(2x)}{2} \right) = 2x + \sin(2x)$$

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

$$= \cos(2x)\cos(x) + (2x + \sin(2x))\sin(x)$$

$$= \cos(2x)\cos(x) + 2x\sin(x) + \sin(2x)\sin(x).$$

$$\begin{aligned} \uparrow \\ \cos(u-v) &= \cos(u)\cos(v) + \sin(u)\sin(v) \quad \perp \end{aligned}$$

$$= \cos(x) + 2x\sin(x)$$

↳ absorbed in y_h

$$= 2x\sin(x).$$

5. Solve the system of differential equations $\frac{dx}{dt} = 2x - y$ by elimination. (20 points)

$$\frac{dx}{dt} - 2x + y = 0$$

$$\frac{dy}{dt} - x = 0$$

$$\Rightarrow \begin{aligned} (D-2)x + y &= 0 \\ -x + Dy &= 0 \end{aligned}$$

$$x = Dy$$

$$(D-2)Dy + y = 0$$

$$(D^2 - 2D + 1)y = 0$$

$$(D-1)^2 y = 0$$

$$\text{Assume } y(t) = e^{rt}$$

$$\Rightarrow (r-1)^2 = 0$$

$$r = 1, 1.$$

$$y(t) = c_1 e^t + c_2 t e^t$$

$$x(t) = y'(t) = c_1 e^t + c_2 e^t + c_2 t e^t$$

$$\begin{cases} x(t) = (c_1 + c_2) e^t + c_2 t e^t \\ y(t) = c_1 e^t + c_2 t e^t \end{cases}$$

Bonus. Find the eigenvalues and eigenfunctions of the boundary value problem

$$y'' + \lambda y = 0, \quad y(0) = 0 \quad y(\pi) = 0. \quad \text{Show all three cases.}$$

(10 points)

$$y(x) = e^{rx}$$

$$\Rightarrow r^2 + \lambda = 0$$

$$r = \pm \sqrt{\lambda}$$

Case 1. $\lambda = 0 \quad r = 0, 0.$

$$y(x) = c_1 + c_2 x$$

$$y(0) = c_1 + 0 = 0 \Rightarrow c_1 = 0$$

$$y(\pi) = c_2 \pi = 0 \Rightarrow c_2 = 0$$

Only trivial solution
satisfies BCs.

Case 2: $\lambda < 0$. Let $\lambda = -\alpha^2$ where $\alpha > 0 \Rightarrow r = \pm \alpha$.

$$y(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)$$

$$y(0) = c_1 + 0 = 0 \Rightarrow c_1 = 0$$

$$y(\pi) = c_2 \sin(\alpha \pi) = 0 \Rightarrow c_2 = 0$$

Only trivial solution
satisfies BCs

Case 3: $\lambda > 0$. Let $\lambda = \alpha^2$ where $\alpha > 0 \Rightarrow r = \pm \alpha i$

$$y(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$$

$$y(0) = c_1 + 0 = 0 \Rightarrow c_1 = 0$$

$$y(\pi) = c_2 \sin(\alpha \pi) = 0$$

$$\sin(\alpha \pi) = 0 \Rightarrow \alpha_n = n, \quad n = 1, 2, 3, \dots$$

$\Rightarrow \lambda_n = \alpha_n^2 = n^2, \quad n = 1, 2, 3, \dots$ are the positive eigenvalues.

$\Rightarrow y_n(x) = \sin(nx), \quad n = 1, 2, 3, \dots$ are the corresponding eigenfunctions.