MTH 204
Fall 2008
Exam 3
Sections A & C

Name: Key

Section: A or C (circle one)

Read the directions carefully.
Write neatly in pencil and show all your work
(you will only receive credit for what you put on your paper).
Please do not share calculators during the test.
Each question is worth 20 points.
DO NO USE decimals on any intermediate step.
The last page contains your Laplace tables.
If you have trouble during the test, feel free to ask me for help.

Score:________

1. Solve the integrodifferential equation $y'(t) = 1 - \int_0^t e^{-2\tau} y(t-\tau)\,d\tau, \quad y(0) = 1.$

$y'(t) = 1 - (e^{-2t} * y(t)), \quad y(0) = 1$

1. Take the Laplace of both sides

$\mathcal{L}\{y'(t)\} = \mathcal{L}\{1\} - \underbrace{\mathcal{L}\{e^{-2t} * y(t)\}}_{\mathcal{L}\{e^{-2t}\}\mathcal{L}\{y(t)\}}$

$(sY(s) - \underset{\underset{1}{\shortparallel}}{y(0)}) = \frac{1}{s} - \frac{1}{s+2} Y(s)$

2. Solve for $Y(s)$

$\left(s + \frac{1}{s+2}\right) Y(s) = 1 + \frac{1}{s} = \frac{s+1}{s}$

$\left(\frac{s(s+2)+1}{s+2}\right) Y(s) = \frac{s+1}{s}$

$Y(s) = \left(\frac{s+1}{s}\right) \frac{(s+2)}{(s+1)^2} = \frac{s+2}{s(s+1)}$

3. Partial Fractions

$\frac{s+2}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{2}{s} - \frac{1}{s+1}$

$s+2 = A(s+1) + Bs$

$s=0 \Rightarrow 2 = A + 0$

$s=-1 \Rightarrow 1 = 0 - B \Rightarrow B = -1$

4. Take the inverse Laplace

$y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$= 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$

$= 2 - e^{-t}$

2. Consider the IVP $y'' - 2y' + y = \delta(t-3), \quad y(0) = 0, y'(0) = 1.$

A. Solve the IVP.

1. Take the Laplace

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{\delta(t-3)\}$$

$$(s^2Y(s) - s\underset{0}{\underset{\parallel}{y(0)}} - \underset{1}{\underset{\parallel}{y'(0)}}) - 2(sY(s) - \underset{0}{\underset{\parallel}{y(0)}}) + Y(s) = e^{-3s}$$

$$(s^2 - 2s + 1)Y(s) - 1 = e^{-3s}$$

2. Solve for Y(s)

$$Y(s) = \frac{1}{(s-1)^2} + \frac{e^{-3s}}{(s-1)^2} = F(s) + e^{-3s}F(s)$$

where $F(s) = \frac{1}{(s-1)^2}$

3. Find f(t)

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = te^t$$

$$f(t-3) = (t-3)e^{t-3}$$

4. Take the inverse Laplace

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{F(s)\} + \mathcal{L}^{-1}\{e^{-3s}F(s)\}$$

$$= f(t) + f(t-3)\mathcal{U}(t-3)$$

$$= te^t + (t-3)e^{t-3}\,\mathcal{U}(t-3)$$

$$= \begin{cases} te^t, & 0 \le t < 3 \\ te^t + (t-3)e^{t-3}, & t \ge 3 \end{cases}$$

B. Evaluate the following.

$y(1) =$ e

$y(4) =$ $4e^4 + e$

3. Solve $\mathbf{x}' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \mathbf{x}$.

1. Find λ's

$0 = \det(A - \lambda I) = \lambda^2 - \mathrm{Tr}A\lambda + \det A = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$

$\Rightarrow \lambda = 1, 1$

2. Find $\vec{K}$

For $\lambda = 1$, $(A - \lambda I)\vec{K} = \vec{0}$

$\begin{bmatrix} 3-1 & -4 \\ 1 & -1-1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} \cancel{2} & \cancel{-4} & \cancel{0} \\ 1 & -2 & 0 \end{array}\right]$ $R_1 = 2R_2$

$u_1 - 2u_2 = 0$ $\Rightarrow \vec{K} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2u_2 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$u_1 = 2u_2$

$\hookrightarrow FV = 1$

$\Rightarrow \vec{X}_1(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t$

3. Find $\vec{P}$

For $\lambda = 1$, $(A - \lambda I)\vec{P} = \vec{K}$

$\begin{bmatrix} 3-1 & -4 \\ 1 & -1-1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} \cancel{2} & \cancel{-4} & \cancel{2} \\ 1 & -2 & 1 \end{array}\right]$ $R_1 = 2R_2$

$p_1 - 2p_2 = 1$ $\Rightarrow \vec{P} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 2p_2 + 1 \\ p_2 \end{bmatrix} = \overset{0}{\cancel{p_2}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$p_1 = 2p_2 + 1$

$\hookrightarrow FV$

$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Rightarrow \vec{X}_2(t) = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^t$

4. Find G.S.

$\vec{x}(t) = c_1 \vec{X}_1(t) + c_2 \vec{X}_2(t)$

$= c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^t$

4. Solve the IVP $\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

1. Find λ's

$0 = \det(A - \lambda I) = \lambda^2 - \mathrm{Tr}A\,\lambda + \det A = \lambda^2 - 2\lambda + 5$

$\Rightarrow \lambda = \dfrac{2 \pm \sqrt{4 - 4(1)(5)}}{2} = \dfrac{2 \pm 4i}{2} = 1 \pm 2i$

Use $\lambda = 1 + 2i$

2. Find $\vec{K}$

For $\lambda = 1 + 2i$, $(A - \lambda I)\vec{K} = \vec{0}$

$\begin{bmatrix} 3-(1+2i) & -2 \\ 4 & -1-(1+2i) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 2(1-i) & -2 & 0 \\ 4 & -2(1+i) & 0 \end{array}\right]$

$R_2 = (1+i)R_1$

$2(1-i)u_1 - 2u_2 = 0$

$u_2 = (1-i)\underbrace{u_1}_{\text{FV}=1}$

$\Rightarrow \vec{K} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ (1-i)u_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

3. Find $\vec{z}$

$\vec{z} = \vec{K}e^{\lambda t} = \vec{K}e^{(1+2i)t} = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) e^{t} \underline{(\cos(2t) + i\sin(2t))}$

$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}\cos(2t) + i \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}\sin(2t) + i \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{t}\cos(2t) - \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{t}\sin(2t$

4. Find $\vec{x}_1, \vec{x}_2$

$\vec{x}_1(t) = \mathrm{Re}\,\vec{z} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}\cos(2t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{t}\sin(2t)$

$\vec{x}_2(t) = \mathrm{Im}\,\vec{z} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}\sin(2t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{t}\cos(2t)$

GS: $\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2$

5. IC

$\vec{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \Rightarrow \begin{array}{l} c_1 = 1 \\ c_1 - c_2 = 3 \Rightarrow c_2 = -2 \end{array}$

$\vec{x}(t) = \vec{x}_1(t) - 2\vec{x}_2(t)$

5. Given that $\Phi(t) = \begin{bmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{bmatrix}$ is a fundamental matrix for $\mathbf{x}' = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \mathbf{x}$, find the general solution to the nonhomogeneous system $\mathbf{x}' = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{2t}$.

1. Find $\vec{x}_h(t)$

$$\vec{x}_h(t) = \Phi(t)\vec{c}$$

2. Find $\Phi^{-1}(t)$

$$\det \Phi(t) = 3 - 1 = 2$$

$$\Phi^{-1}(t) = \frac{1}{2}\begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{t} & 3e^{t} \end{bmatrix}$$

3. Find $\vec{x}_p(t) = \Phi(t)\int \Phi^{-1}(t)\,\vec{F}(t)\,dt$

$$\vec{x}_p(t) = \frac{1}{2}\Phi(t)\int \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{t} & 3e^{t} \end{bmatrix}\begin{bmatrix} 3e^{2t} \\ e^{2t} \end{bmatrix} dt$$

$$= \frac{1}{2}\Phi(t)\int \begin{bmatrix} 3e^{t} - e^{t} \\ -3e^{3t} - 3e^{3t} \end{bmatrix} dt$$

$$= \Phi(t)\int \begin{bmatrix} e^{t} \\ 0 \end{bmatrix} dt$$

$$= \begin{bmatrix} 3e^{t} & e^{-t} \\ e^{t} & e^{-t} \end{bmatrix}\begin{bmatrix} e^{t} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{2t}$$

4. Find GS

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$= \Phi(t)\vec{c} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{2t}$$

Bonus (10 points): The matrix exponential, e^{At}, can be found using the equation $e^{At} = \mathscr{L}^{-1}\{(sI - A)^{-1}\}$. Compute e^{At} for the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. What is the relationship between e^{At} and the homogeneous system $\mathbf{x}' = A\mathbf{x}$?

$$sI - A = \begin{bmatrix} s-1 & -1 \\ 0 & s-1 \end{bmatrix}$$

$$\det(sI - A) = (s-1)^2$$

$$(sI-A)^{-1} = \frac{1}{(s-1)^2}\begin{bmatrix} s-1 & 1 \\ 0 & s-1 \end{bmatrix} = \begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-1)^2} \\ 0 & \frac{1}{s-1} \end{bmatrix}$$

Now $e^{At} = \mathscr{L}^{-1}\{(sI-A)^{-1}\}$

$$= \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$$

e^{At} is a fundamental matrix for $\vec{x}' = A\vec{x}$!