MTH 204
Fall 2006
Exam 3

Name: Key

Section B

Read the directions carefully.
Write neatly in pencil and show all your work (you will only get credit for what you put on paper).
Please do not share calculators during the test.
Each question is worth 20 points
DO NOT USE Decimals on any intermediate step.
The last page contains your Laplace tables.
If you have trouble during the test, feel free to ask me for help.

Score:________

1. Consider the differential equation $x^2y''-4xy'=x^5$

   a. Classify the differential equation by order, linearity, type of coefficients, and state whether or not the equation is homogeneous.

   2nd order, linear, variable coefficients, nonhomogeneous (Cauchy - Euler)

   b. What method(s) can you use to solve this equation?

   $y_h \Rightarrow y=x^m \qquad y_p \Rightarrow \text{VOP}$

   c. Solve the equation for the interval $x>0$.

$x^2y''-4xy'=0 \qquad y=x^m$

$y'=mx^{m-1}$

$y''=m(m-1)x^{m-2}$

$\Rightarrow x^2[m(m-1)x^{m-2}]-4xmx^{m-1}=0$

$x^m[m^2-m-4m]=0$

$m^2-5m=0$

$\Rightarrow m=0,5$

$y_h=c_1+c_2x^5$

Put DE in Standard form; $y''-\frac{4}{x}y'=x^3$

Assume $y_p=u_1(x)y_1(x)+u_2(x)y_2(x)$

$W(y_1,y_2)=\begin{vmatrix} 1 & x^5 \\ 0 & 5x^4 \end{vmatrix}=5x^4$

$W_1=\begin{vmatrix} 0 & x^5 \\ x^3 & 5x^4 \end{vmatrix}=0-x^8=-x^8$

$W_2=\begin{vmatrix} 1 & 0 \\ 0 & x^3 \end{vmatrix}=x^3$

$u_1'(x)=\frac{W_1}{W(y_1,y_2)}=\frac{-x^8}{5x^4}=\frac{-1}{5}x^4 \qquad u_2'(x)=\frac{W_2}{W(y_1,y_2)}=\frac{x^3}{5x^4}=\frac{1}{5x}$

$\Rightarrow u_1(x)=\frac{-1}{25}x^5 \qquad \Rightarrow u_2(x)=\frac{1}{5}\ln x$

$\Rightarrow y_p=\underbrace{\frac{-1}{25}x^5(1)}_{\text{absorbed into } y_2}+\frac{x^5}{5}\ln x=\frac{x^5}{5}\ln x$

Then $y=y_h+y_p=c_1+c_2x^5+\frac{x^5}{5}\ln x$

2. Suppose a 160 lb person stretches a bungee cord on the New River Gorge Bridge 200 ft. Then they are displaced an additional 100 ft before the cord is released. Assume there is no damping.

a. Set up, **but do not solve** the initial value problem describing this motion. **Use** $g = 32 \text{ft}/\text{s}^2$ as the acceleration for gravity.

$$m = \frac{W}{g} = \frac{160}{32} = 5$$

Equilibrium position: $mg = W = Ks$

$$160 = 200K$$

$$\Rightarrow K = \frac{4}{5}$$

No damping $\Rightarrow \beta = 0$

$$5y'' + \frac{4}{5}y = 0 \qquad \begin{cases} y(0) = 100 & \text{initial displacement} \\ y'(0) = 0 & \text{initial velocity.} \end{cases}$$

b. Now assume that the object is acted on by an external force of $f(t) = 20\cos(\gamma t)$. For what value of γ does resonance occur?

Resonance occurs when $\omega = \gamma$ where $\omega = \sqrt{\frac{K}{m}}$

$$= \sqrt{\frac{4/5}{5}}$$

$$= \sqrt{\frac{4}{25}}$$

$$= \frac{2}{5}$$

3. Use the definition of the Laplace transform ($\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$) to find $\mathcal{L}\{f(t)\}$ when $f(t) = \begin{cases} t & 0 \le t < 1 \\ 1 & t \ge 1 \end{cases}$. For what values of s is the Laplace transform of f(t) defined?

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

$$= \int_0^1 e^{-st} t\,dt + \int_1^\infty e^{-st} dt$$

$u = t \qquad dv = e^{-st}$

$du = dt \qquad v = -\frac{1}{s} e^{-st}$

$$= -\frac{t}{s} e^{-st} \Big|_0^1 + \frac{1}{s} \int_0^1 e^{-st} dt \; - \frac{1}{s} e^{-st} \Big|_1^\infty$$

$$= -\left(\frac{1}{s} e^{-s} - 0\right) - \frac{1}{s^2} e^{-st} \Big|_0^1 - \left(0 - \frac{1}{s} e^{-s}\right)$$

$$= -\frac{1}{s} e^{-s} - \left(\frac{1}{s^2} e^{-s} - \frac{1}{s^2}\right) + \frac{1}{s} e^{-s}$$

$$= -\frac{1}{s^2} e^{-s} + \frac{1}{s^2} \quad \text{Provided } s > 0$$

4. Find the inverse Laplace transform of the following:

a. $F(s)=\dfrac{2s-6}{s^2+25}$

$$\mathcal{L}^{-1}\left\{\frac{2s-6}{s^2+25}\right\}=2\mathcal{L}^{-1}\left\{\frac{s}{s^2+25}\right\}-6\mathcal{L}^{-1}\left\{\frac{1}{s^2+25}\right\}$$

$$=2\mathcal{L}^{-1}\left\{\frac{s}{s^2+25}\right\}-6\mathcal{L}^{-1}\left\{\left(\frac{5}{5}\right)\left(\frac{1}{s^2+25}\right)\right\}$$

$$=2\mathcal{L}^{-1}\left\{\frac{s}{s^2+25}\right\}-\frac{6}{5}\mathcal{L}^{-1}\left\{\frac{5}{s^2+25}\right\}$$

$$=2\cos(5t)-\frac{6}{5}\sin(5t)$$

b. $G(s)=\dfrac{s}{s^2-4s+13}$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2-4s+13}\right\}=\mathcal{L}^{-1}\left\{\frac{s}{s^2-4s+4-4+13}\right\}=\mathcal{L}^{-1}\left\{\frac{s}{(s-2)^2+9}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{s-2+2}{(s-2)^2+3^2}\right\}=\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+3^2}\right\}+2\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+3^2}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\bigg|_{s\to s-2}\right\}+2\mathcal{L}^{-1}\left\{\frac{1}{s^2+3^2}\bigg|_{s\to s-2}\right\}$$

$$=e^{2t}\cos(3t)+\frac{2}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3}\bigg|_{s\to s-2}\right\}$$

$$=e^{2t}\cos(3t)+\frac{2}{3}e^{2t}\sin(3t)$$

c. $I(s)=\dfrac{e^{-s}}{s(s+1)}$

$$\frac{1}{s(s+1)}=\frac{A}{s}+\frac{B}{s+1}\implies 1=A(s+1)+Bs$$

$s=0 \quad A=1$

$s=-1 \quad B=-1$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{1}{s+1}\right\}=1-e^{-t}$$

$$\implies \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s+1)}\right\}=\left(1-e^{-(t-1)}\right)u(t-1)$$

5. Use the Laplace transform to solve the initial value problem $y''+y=\delta(t-\frac{3\pi}{2})$ subject to $y(0)=y'(0)=0$.

$$\mathcal{L}\{y''\}+\mathcal{L}\{y\}=\mathcal{L}\{\delta(t-\tfrac{3\pi}{2})\}$$

$$s^2Y(s)-s\underset{0}{\underset{\parallel}{y(0)}}-\underset{0}{\underset{\parallel}{y'(0)}}+Y(s)=e^{-\frac{3\pi}{2}s}$$

$$(s^2+1)Y(s)=e^{-\frac{3\pi}{2}s}$$

$$Y(s)=\frac{e^{-\frac{3\pi}{2}s}}{s^2+1}$$

$$F(s)=\frac{1}{s^2+1} \implies f(t)=\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}=\sin(t)$$

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\{e^{-\frac{3\pi}{2}s}F(s)\} \\ &= \sin(t-\tfrac{3\pi}{2})\mathcal{U}(t-\tfrac{3\pi}{2}) \\ &= \cos(t)\,\mathcal{U}(t-\tfrac{3\pi}{2}) \end{aligned}$$

Bonus (15 points): Use the Laplace transform to solve the initial value problem

$y'+y = \mathcal{U}(t-1)$ subject to $y(0) = 1$. Graph your solution.

$$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{\mathcal{U}(t-1)\}$$

$$sY(s) - \underset{1}{\underset{\shortparallel}{y(0)}} + Y(s) = \frac{e^{-s}}{s}$$

$$(s+1)Y(s) - 1 = \frac{e^{-s}}{s}$$

$$(s+1)Y(s) = 1 + \frac{e^{-s}}{s}$$

$$Y(s) = \frac{1}{s+1} + \frac{e^{-s}}{s(s+1)}$$

$$F(s) = \frac{1}{s(s+1)}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s+1)}\right\}$$

$$= e^{-t} + \left(1 - e^{-(t-1)}\right)\mathcal{U}(t-1) \quad \text{from 4c}$$

$$= \begin{cases} e^{-t}, & 0 \le t < 1 \\ e^{-t} + 1 - e^{-(t-1)}, & t \ge 1 \end{cases}$$