MTH 204
Spring 2007
Exam 3

Name: Key

Section: B or C (circle one)

Read the directions carefully.
Write neatly in pencil and show all your work (you will only get credit for what you put on paper).
Please do not share calculators during the test.
Each question is worth 20 points
DO NOT USE Decimals on any intermediate step.
The last page contains your Laplace tables.
If you have trouble during the test, feel free to ask me for help.

Score:________

1. Consider the differential equation $x^2y''+2xy'-6y = x^2$.

   a. Classify the differential equation by order, linearity, type of coefficients, and state whether or not the equation is homogeneous.

   2nd order, linear, variable coefficients, non homogeneous (Cauchy-Euler)

   b. What method(s) can you use to solve this equation?

   $y_h: y(x) = x^m$ $\qquad$ $y_p: \begin{cases} \text{VOP} \\ x = e^t \Rightarrow \text{MUC} \end{cases}$

   c. Solve the equation for the interval $x > 0$.

$$x^2y''+2xy'-6y=0$$
$$y(x)=x^m$$
$$y'(x)=mx^{m-1}$$
$$y''(x)=m(m-1)x^{m-2}$$
$$x^2(m(m-1)x^{m-2})+2xmx^{m-1}-6x^m=0$$
$$x^m[m^2-m+2m-6]=0$$

$$m^2+m-6=0$$
$$(m-2)(m+3)=0$$
$$m=2,-3$$

$$y_h(x)=c_1x^2+c_2x^{-3}$$

Put DE in standard form: $y''+\frac{2}{x}y'-\frac{6}{x^2}y=1$

Assume $y_p = u_1(x)y_1(x)+u_2(x)y_2(x)$

$$W(y_1,y_2)=\begin{vmatrix} x^2 & x^{-3} \\ 2x & -3x^{-4} \end{vmatrix} = -3x^2x^{-4}-2xx^{-3} = -5x^{-2}$$

$$W_1=\begin{vmatrix} 0 & x^{-3} \\ 1 & -3x^{-4} \end{vmatrix} = -x^{-3} \qquad W_2=\begin{vmatrix} x^2 & 0 \\ 2x & 1 \end{vmatrix} = x^2$$

$$u_1'(x)=\frac{W_1}{W(y_1,y_2)}=\frac{-x^{-3}}{-5x^{-2}}=\frac{1}{5}x^{-1} \qquad u_2'(x)=\frac{W_2}{W(y_1,y_2)}=\frac{x^2}{-5x^{-2}}=-\frac{1}{5}x^4$$

$$u_1(x)=\frac{1}{5}\int x^{-1}dx=\frac{1}{5}\ln x \qquad u_2(x)=-\frac{1}{5}\int x^4dx=-\frac{1}{25}x^5$$

$$\Rightarrow y_p(x)=u_1y_1+u_2y_2=\left(\frac{1}{5}\ln x\right)x^2-\frac{1}{25}x^5x^{-3}=\frac{1}{5}x^2\ln x-\underbrace{\frac{1}{25}x^2}_{\text{absorbed into } y_h}=\frac{1}{5}x^2\ln x$$

So $y(x)=y_h+y_p=c_1x^2+c_2x^{-3}+\frac{1}{5}x^2\ln x$

2. Suppose a 4 lb object in a tank of water stretches a spring 2 ft to equilibrium position. The viscosity of the water offers a resistance to the motion of the object numerically equal to the instantaneous velocity. Assume there is no external force. Assume the object is released from 1 ft above the equilibrium position with a downward velocity of 8 ft/s.

a. Set up the IVP describing this motion. **Use** $g = 32\text{ft}/s^2$ as the acceleration for gravity.

$m = \frac{W}{g} = \frac{4}{32} = \frac{1}{8}$

$\beta = 1$

$K = \frac{W}{s} = \frac{4}{2} = 2$

$f(x) = 0$

$my'' + \beta y' + Ky = f(x)$

$\frac{1}{8}y'' + y' + 2y = 0$

$y'' + 8y' + 16y = 0$

$\begin{cases} y(0) = -1 \\ y'(0) = 8 \end{cases}$

b. Solve for the displacement y(t).

Assume $y(t) = e^{rt}$

$r^2 + 8r + 16 = 0$

$(r+4)^2 = 0$

$r = -4, -4$

$y(t) = c_1 e^{-4t} + c_2 t e^{-4t}$

$y(0) = c_1 + 0 = -1$

$y(t) = -e^{-4t} + c_2 t e^{-4t}$

$y'(t) = 4e^{-4t} + c_2 e^{-4t} - 4c_2 t e^{-4t}$

$y'(0) = 4 + c_2 - 0 = 8$

$\Rightarrow c_2 = 4$

$y(t) = -e^{-4t} + 4te^{-4t}$

c. Find the time at which the object attains its extreme displacement from the equilibrium position. (**Hint:** when do you have a local max/min?)

$y'(t) = 4e^{-4t} + 4e^{-4t} - 16te^{-4t} = 0$

$e^{-4t}(8 - 16t) = 0$

$t = \frac{8}{16} = \frac{1}{2}\text{s}$

3. Use the definition of the Laplace transform ($\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt$) to find $\mathcal{L}\{f(t)\}$

when $f(t) = \begin{cases} t & 0 \le t < 1 \\ 1 & t \ge 1 \end{cases}$. For what values of s is the Laplace transform of f(t) defined?

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt$$

$$= \int_0^1 e^{-st} t\,dt + \int_1^{\infty} e^{-st} dt$$

$u = t \qquad dv = e^{-st}$

$du = dt \qquad v = -\frac{1}{s}e^{-st}$

$$= -\frac{t}{s}e^{-st}\Big|_0^1 - \left(-\frac{1}{s}\right)\int_0^1 e^{-st}dt + \int_1^{\infty} e^{-st}dt$$

$$= -\frac{1}{s}e^{-s} + 0 + \frac{1}{s}\left(-\frac{1}{s}e^{-st}\Big|_0^1\right) - \frac{1}{s}e^{-st}\Big|_1^{\infty}$$

$$= -\frac{1}{s}e^{-s} + \frac{1}{s}\left(-\frac{1}{s}e^{-s} + \frac{1}{s}\right) - 0 + \frac{1}{s}e^{-s}$$

$$= \frac{1}{s^2} - \frac{1}{s^2}e^{-s}, \text{ provided } s > 0$$

4. Find the inverse Laplace transform of the following:

a. $F(s) = \dfrac{s}{s^2 - 4s + 13}$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2-4s+13}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2-4s+\left(\frac{4}{2}\right)^2-\left(\frac{4}{2}\right)^2+13}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{(s-2)^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{s-2+2}{(s-2)^2+9}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+9}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+9}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\Big|_{s\to s-2}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\Big|_{s\to s-2}\right\}$$

$$= e^{2t}\cos(3t) + \frac{2}{3}e^{2t}\sin(3t)$$

b. $G(s) = \dfrac{e^{-2s}}{s^2(s-1)}$

$$H(s) = \frac{1}{s^2(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} = \frac{-1}{s} - \frac{1}{s^2} + \frac{1}{s-1}$$

$$1 = As(s-1) + B(s-1) + Cs^2$$

$s=0 \Rightarrow B=-1$

$s=1 \Rightarrow C=1$

$s=-1 \Rightarrow 1 = 2A-(1)(-2)+1 \Rightarrow A=-1$

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = -\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = -1-t+e^t$$

$$g(t) = \mathcal{L}^{-1}\{e^{-2s}H(s)\} = (-1-(t-2)+e^{t-2})\mathcal{U}(t-2)$$

$$= (1-t+e^{t-2})\mathcal{U}(t-2)$$

5. Consider the IVP $y''+y=\delta(t-\frac{\pi}{2})+\delta(t-2\pi)$ subject to $y(0)=1, y'(0)=0$.

a. Solve for y(t).

$$\mathcal{L}\{y''+y\}=\mathcal{L}\{\delta(t-\tfrac{\pi}{2})+\delta(t-2\pi)\}$$

$$\mathcal{L}\{y''\}+\mathcal{L}\{y\}=\mathcal{L}\{\delta(t-\tfrac{\pi}{2})\}+\mathcal{L}\{\delta(t-2\pi)\}$$

$$s^2Y(s)-s\underset{1}{\underset{\parallel}{y(0)}}-\underset{0}{\underset{\parallel}{y'(0)}}+Y(s)=e^{-\frac{\pi}{2}s}+e^{-2\pi s}$$

$$(s^2+1)Y(s)=e^{-\frac{\pi}{2}s}+e^{-2\pi s}+s$$

$$Y(s)=\frac{e^{-\frac{\pi}{2}s}}{s^2+1}+\frac{e^{-2\pi s}}{s^2+1}+\frac{s}{s^2+1}=e^{-\frac{\pi}{2}s}F(s)+e^{-2\pi s}F(s)+\frac{s}{s^2+1}$$

Let $F(s)=\frac{1}{s^2+1} \Rightarrow f(t)=\sin(t)$

$$y(t)=\mathcal{L}^{-1}\{Y(s)\}$$

$$=\mathcal{L}^{-1}\{e^{-\frac{\pi}{2}s}F(s)\}+\mathcal{L}^{-1}\{e^{-2\pi s}F(s)\}+\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$$

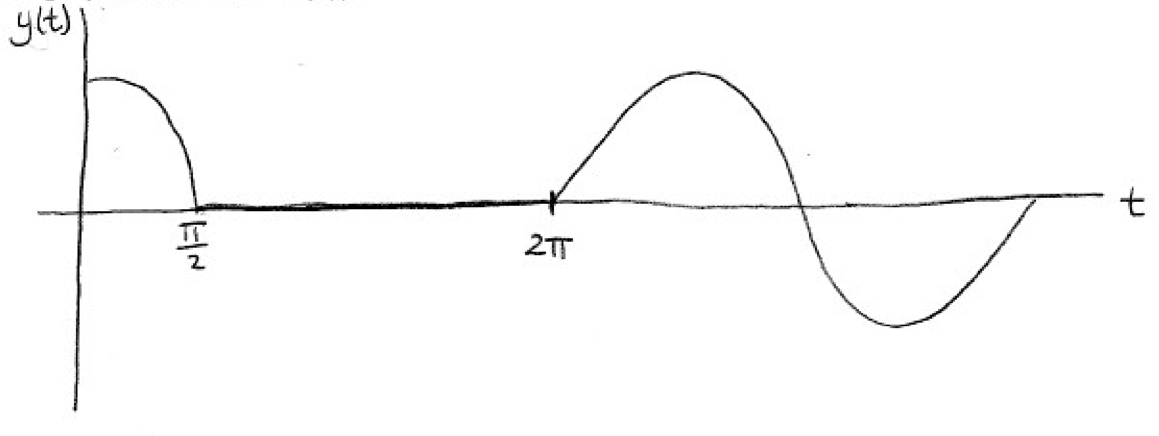
$$=\sin(t-\tfrac{\pi}{2})\mathcal{U}(t-\tfrac{\pi}{2})+\sin(t-2\pi)\mathcal{U}(t-2\pi)+\cos(t)$$

$$=-\cos(t)\mathcal{U}(t-\tfrac{\pi}{2})+\sin(t)\mathcal{U}(t-2\pi)+\cos(t)$$

or

$$y(t)=\begin{cases}\cos(t), & 0\le t<\pi/2\\ -\cos(t)+\cos(t)=0, & \frac{\pi}{2}\le t<2\pi\\ \sin(t), & t\ge 2\pi.\end{cases}$$

b. Graph your solution for y(t).



Bonus (10 points): Use the Laplace transform to solve the IVP $y' + y = \mathcal{U}(t-1)$ subject to $y(0) = 1$. Graph your solution.

$$\mathcal{L}\{y' + y\} = \mathcal{L}\{\mathcal{U}(t-1)\}$$

$$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{\mathcal{U}(t-1)\}$$

$$sY(s) - \underbrace{y(0)}_{=1} + Y(s) = \frac{e^{-s}}{s}$$

$$(s+1)Y(s) = \frac{e^{-s}}{s} + 1$$

$$Y(s) = \frac{e^{-s}}{s(s+1)} + \frac{1}{s+1}$$

Let $F(s) = \dfrac{1}{s(s+1)} = \dfrac{A}{s} + \dfrac{B}{s+1} = \dfrac{1}{s} - \dfrac{1}{s+1}$

$1 = A(s+1) + Bs$

$s = 0 \Rightarrow A = 1$

$s = -1 \Rightarrow B = -1$

$f(t) = 1 - e^{-t}$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{e^{-s}F(s)\} + \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= (1 - e^{-(t-1)})\,\mathcal{U}(t-1) + e^{-t}$$

or $y(t) = \begin{cases} e^{-t}, & 0 \le t < 1 \\ 1 - e^{-(t-1)} + e^{-t}, & t \ge 1 \end{cases}$