MTH 204
Spring 2008
Exam 3

Math 204
Exam 3
Sections C&F
Spring 2008

Name: Key

Section: C or F (circle one)

Read the directions carefully.
Write neatly in pencil and show all your work
(you will only get credit for what you put on paper).
Each question is worth 20 points.
DO NOT use decimals in any intermediate step
Please do not share calculators during the test.
The last page contains your Laplace tables.
If you have trouble during the test, feel free to ask me for help.

Score:________

1. Solve the integrodifferential equation $y'(t)+3\int_0^t \cos(\tau)y(t-\tau)d\tau=\sin(t)$, y(0) = 0.

1. Take Laplace transform of both sides

$$\mathcal{L}\{y'(t)\}+3\mathcal{L}\{\cos(t)*y(t)\}=\mathcal{L}\{\sin(t)\}$$

$$sY(s)-\overset{=0}{y(0)}+\frac{3sY(s)}{s^2+1}=\frac{1}{s^2+1}$$

2. Solve for Y(s)

$$\left(s+\frac{3s}{s^2+1}\right)Y(s)=\left(\frac{s(s^2+4)}{s^2+1}\right)Y(s)=\frac{1}{s^2+1}$$

$$\Rightarrow Y(s)=\frac{1}{s^2+1}\left(\frac{s^2+1}{s(s^2+4)}\right)=\frac{1}{s(s^2+4)}$$

3. Partial Fractions

$$\frac{1}{s(s^2+4)}=\frac{A}{s}+\frac{Bs+C}{s^2+4}\Rightarrow 1=A(s^2+4)+Bs^2+Cs$$

$$\begin{cases} s=0 \Rightarrow 4A=1 \Rightarrow A=\frac{1}{4} \\ s=2i \Rightarrow 1+0i=-4B+2iC \Rightarrow B=\frac{-1}{4} \\ \qquad C=0 \end{cases}$$

4. Take inverse Laplace transform of both sides

$$y(t)=\mathcal{L}^{-1}\{Y(s)\}$$

$$=\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}$$

$$=\frac{1}{4}-\frac{1}{4}\cos(2t)$$

2. Consider the IVP $y''+y=\delta(t-\frac{\pi}{2})+\delta(t-\frac{3\pi}{2})$, $y(0)=y'(0)=0$.

A. Solve the IVP.

1. Take the Laplace transform of both sides

$$\mathcal{L}\{y''\}+\mathcal{L}\{y\}=\mathcal{L}\{\delta(t-\tfrac{\pi}{2})\}+\mathcal{L}\{\delta(t-\tfrac{3\pi}{2})\}$$

$$s^2Y(s)-s\overbrace{y(0)}^{=0}-\overbrace{y'(0)}^{=0}+Y(s)=e^{-\frac{\pi}{2}s}+e^{-\frac{3\pi}{2}s}$$

2. Solve for $Y(s)$

$$(s^2+1)Y(s)=e^{-\frac{\pi}{2}s}+e^{-\frac{3\pi}{2}s}$$

$$\Rightarrow Y(s)=\frac{e^{-\frac{\pi}{2}s}}{s^2+1}+\frac{e^{-\frac{3\pi}{2}s}}{s^2+1}=e^{-\frac{\pi}{2}s}F(s)+e^{-\frac{3\pi}{2}s}F(s)$$

where $F(s)=\frac{1}{s^2+1}$

3. Find $f(t)$

$f(t)=\sin(t)$

4. Take the inverse Laplace of both sides

$$y(t)=\mathcal{L}^{-1}\{e^{-\pi/2\,s}F(s)\}+\mathcal{L}^{-1}\{e^{-\frac{3\pi}{2}s}F(s)\}$$

$$=f(t-\tfrac{\pi}{2})\mathcal{U}(t-\pi/2)+f(t-\tfrac{3\pi}{2})\mathcal{U}(t-\tfrac{3\pi}{2})$$

$$=\sin(t-\tfrac{\pi}{2})\mathcal{U}(t-\tfrac{\pi}{2})+\sin(t-\tfrac{3\pi}{2})\mathcal{U}(t-\tfrac{3\pi}{2})$$

$$=-\cos(t)\mathcal{U}(t-\pi/2)+\cos(t)\mathcal{U}(t-\tfrac{3\pi}{2})$$

$$=\begin{cases}0, & 0\le t<\pi/2\\ -\cos(t), & \pi/2\le t<3\pi/2\\ -\cos(t)+\cos(t), & t\ge 3\pi/2\end{cases}$$

B. Evaluate the following.

$y(\frac{\pi}{4})=$ 0

$y(\pi)=$ $-\cos(\pi)=-(-1)=1$

$y(\frac{5\pi}{2})=$ 0

3. Solve the IVP $\vec{x}\,' = \begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix}\vec{x},\ \vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

1. Find λ

$$0 = \det(A-\lambda I) = \lambda^2 - (6+2)\lambda + 16 = (\lambda-4)^2 \Rightarrow \lambda = 4,4$$

2. Find $\vec{K}$

For $\lambda = 4$, $(A-\lambda I)\vec{K} = \vec{0}$

$$\left[\begin{array}{cc|c} 2-4 & 4 & 0 \\ -1 & 6-4 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} \cancel{-2} & \cancel{4} & \cancel{0} \\ -1 & 2 & 0 \end{array}\right] \Rightarrow \begin{aligned} -u_1 + 2u_2 &= 0 \\ u_1 &= 2u_2 \end{aligned}$$

$\leftarrow$ FV $= 1$

$$\Rightarrow \vec{K} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2u_2 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \Rightarrow \vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}$$

3. Find $\vec{P}$

For $\lambda = 4$, $(A-\lambda I)\vec{P} = \vec{K}$

$$\left[\begin{array}{cc|c} 2-4 & 4 & 2 \\ -1 & 6-4 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} \cancel{-2} & \cancel{4} & \cancel{2} \\ -1 & 2 & 1 \end{array}\right] \Rightarrow \begin{aligned} -p_1 + 2p_2 &= 1 \\ p_1 &= 2p_2 - 1 \end{aligned}$$

$\leftarrow$ FV $= 0$

$$\Rightarrow \vec{P} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 2p_2 - 1 \\ p_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad \Rightarrow \vec{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^{4t} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{4t}$$

4. GS: $\vec{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t} + c_2\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) e^{4t}$

5. IC: $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$$\Rightarrow c_1 = 0,\ c_2 = -1$$

$$\vec{x}(t) = -\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) e^{4t}$$

4. Solve $\vec{x}' = \begin{bmatrix} 6 & -1 \\ 5 & 2 \end{bmatrix} \vec{x}$.

1. Find λ

$$0 = \det(A - \lambda I) = \lambda^2 - (6+2)\lambda + 17 = \lambda^2 - 8\lambda + 17$$

$$\Rightarrow \lambda = \frac{8 \pm \sqrt{64 - 4(17)}}{2} = \frac{8 \pm 2i}{2} = 4 \pm i, \text{ use } \lambda = 4+i$$

2. Find $\vec{K}$

For $\lambda = 4+i$, $(A - \lambda I)\vec{K} = \vec{0}$

$$\left[\begin{array}{cc|c} 6-(4+i) & -1 & 0 \\ 5 & 2-(4+i) & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 2-i & -1 & 0 \\ \cancel{5} & \cancel{-(2+i)} & \cancel{0} \end{array}\right] \Rightarrow \begin{array}{l} (2-i)u_1 - u_2 = 0 \\ u_2 = (2-i)u_1 \end{array}$$

FV $= 1$

$$\Rightarrow \vec{K} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ (2-i)u_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2-i \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

3. Find $\vec{z}$

$$\vec{z} = \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) e^{(4+i)t} = \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) e^{4t}(\cos(t) + i\sin(t))$$

$$= e^{4t}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\cos(t) + i\begin{bmatrix} 1 \\ 2 \end{bmatrix}\sin(t) + i\begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos(t) - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin(t) \right)$$

$$\vec{x}_1 = \operatorname{Re}\vec{z} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{4t}\cos(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{4t}\sin(t) = \begin{bmatrix} \cos(t) \\ 2\cos(t) + \sin(t) \end{bmatrix} e^{4t}$$

$$\vec{x}_2 = \operatorname{Im}\vec{z} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{4t}\sin(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{4t}\cos(t) = \begin{bmatrix} \sin(t) \\ 2\sin(t) - \cos(t) \end{bmatrix} e^{4t}$$

4. GS: $\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2$

$$= c_1 \begin{bmatrix} \cos(t) \\ 2\cos(t) + \sin(t) \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} \sin(t) \\ 2\sin(t) - \cos(t) \end{bmatrix} e^{4t}$$

5. Given that $\Phi(t)=\begin{bmatrix} 3e^{t} & e^{-t} \\ e^{t} & e^{-t} \end{bmatrix}$ is a fundamental matrix for $\bar{x}'=\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}\bar{x}$, find the general solution to the nonhomogeneous system $\bar{x}'=\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}\bar{x}+\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t}$.

1. Find $\Phi(t)=\begin{bmatrix} 3e^{t} & e^{-t} \\ e^{t} & e^{-t} \end{bmatrix}$

$\Rightarrow \vec{x}_h=\Phi(t)\vec{c}$

2. Find $\Phi^{-1}(t)$

$\det\Phi(t)=3-1=2$

$\Rightarrow \Phi^{-1}(t)=\frac{1}{2}\begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{t} & 3e^{t} \end{bmatrix}$.

3. Find $\vec{x}_p$

$$\vec{x}_p=\Phi(t)\int\Phi^{-1}(t)\vec{F}(t)dt=\frac{1}{2}\Phi(t)\int\begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{t} & 3e^{t} \end{bmatrix}\begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}dt$$

$$=\frac{1}{2}\Phi(t)\int\begin{bmatrix} e^{t}-e^{t} \\ -e^{3t}+3e^{3t} \end{bmatrix}dt=\frac{1}{2}\Phi(t)\int\begin{bmatrix} 0 \\ 2e^{3t} \end{bmatrix}dt$$

$$=\frac{1}{2}\begin{bmatrix} 3e^{t} & e^{-t} \\ e^{t} & e^{-t} \end{bmatrix}\begin{bmatrix} 0 \\ \frac{2}{3}e^{3t} \end{bmatrix}=\frac{1}{2}\begin{bmatrix} 2/3\,e^{2t} \\ 2/3\,e^{2t} \end{bmatrix}=\frac{1}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t}$$

4. GS: $\vec{x}(t)=\vec{x}_h+\vec{x}_p$

$$=\begin{bmatrix} 3e^{t} & e^{-t} \\ e^{t} & e^{-t} \end{bmatrix}\vec{c}+\frac{1}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t}$$

Bonus (10 points): Use the Laplace transform to solve the given system of differential equations.

$$\frac{d^2x}{dt^2}+3\frac{dy}{dt}+3y=0$$
, where x(0) = x'(0) = y(0) = 0.
$$\frac{d^2x}{dt^2}+3y=e^{-t}$$

1. Take Laplace of both equations

$$\begin{cases} \mathcal{L}\{x''\}+3\mathcal{L}\{y'\}+3\mathcal{L}\{y\}=\mathcal{L}\{0\} \\ \mathcal{L}\{x''\}+3\mathcal{L}\{y\}=\mathcal{L}\{e^{-t}\} \end{cases}$$

$$\begin{cases} s^2X(s)-s\overset{=0}{x(0)}-\overset{=0}{x'(0)}+3(sY(s)-\overset{=0}{y(0)})+3Y(s)=0 \\ s^2X(s)-sx(0)-x'(0)+3Y(s)=\frac{1}{s+1} \end{cases}$$

$$\begin{array}{r} \\ -(s+1) \end{array}\begin{cases} s^2X(s)+3(s+1)Y(s)=0 \\ s^2X(s) \qquad +3Y(s)=\frac{1}{s+1} \end{cases}$$

2. Solve for X(s)

$$[s^2-s^2(s+1)]X(s)=-1 \Rightarrow s^2[1-s-1]X(s)=-1$$
$$\Rightarrow s^3X(s)=1$$
$$\Rightarrow X(s)=\frac{1}{s^3}$$

3. Find x(t)

$$x(t)=\mathcal{L}^{-1}\{X(s)\}=\frac{1}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{s^{2+1}}\right\}=\frac{1}{2}t^2$$

4. Find y(t)

$$y(t)=\frac{1}{3}(e^{-t}-x''(t))=\frac{1}{3}(e^{-t}-1)$$

Sol: $\begin{cases} x(t)=\frac{1}{2}t^2 \\ y(t)=\frac{1}{3}(e^{-t}-1). \end{cases}$