MTH 204
Spring 2006
Exam 3

Name: Key

Section: D/E

Math 204
Sections D&E
Exam 3 Spring 2006

Read the directions carefully.
Write neatly in pencil and show all your work (you will only get credit for what you put on paper).
Please do not share calculators during the test.
Each question is worth 20 points
DO NOT USE Decimals in any intermediate step.
The last page contains your Laplace tables.
If you have trouble during the test, feel free to ask me for help.

Score:________

1. Consider the differential equation $x^2y''-xy'+y=2x$

   a. Classify the differential equation by order, linearity, type of coefficients, and state whether or not the equation is homogeneous.

   2nd order, linear, variable coefficients, & nonhomogeneous
   (Cauchy-Euler)

   b. What method(s) can you use to solve this equation?

   $y_h \Rightarrow$ Assume $y=x^m$ $\qquad$ $y_p: \begin{cases} x=e^t \Rightarrow \text{MUC} \\ \text{VOP} \end{cases}$

   c. Solve the equation for the interval $x>0$.

$x^2y''-xy'+y=0$ $\quad$ Assume $y(x)=x^m$

$x^2[m(m-1)x^{m-2}]-x[mx^{m-1}]+x^m=0$

$x^m[m^2-m-m+1]=0$

$\Rightarrow m^2-2m+1=0$

$(m-1)^2=0$

$m_1=m_2=1$

$\Rightarrow \left.\begin{array}{l} y_1=x \\ y_2=x\ln x \end{array}\right\} y_h=c_1x+c_2x\ln x$

Method 1: VOP

$y''-\frac{1}{x}y'+\frac{1}{x^2}y=\frac{2}{x}$

$W(y_1,y_2)=\begin{vmatrix} x & x\ln x \\ 1 & \ln x+1 \end{vmatrix} = x\ln x+x-x\ln x=x$

$W_1=\begin{vmatrix} 0 & x\ln x \\ 2x^{-1} & \ln x+1 \end{vmatrix} = -2\ln x$

$W_2=\begin{vmatrix} x & 0 \\ 1 & 2x^{-1} \end{vmatrix} = 2$

$u_1'=\frac{W_1}{W}=\frac{-2\ln x}{x} \Rightarrow u_1=-2\int\frac{\ln x}{x}dx = -2\int s\,ds=-s^2=-(\ln x)^2$

$s=\ln x \quad ds=x^{-1}dx$

$u_2'=\frac{W_2}{W}=\frac{2}{x} \Rightarrow u_2=2\int\frac{dx}{x}=2\ln x$

$y_p=u_1y_1+u_2y_2=-(\ln x)^2x+2(\ln x)(x\ln x)=x(\ln x)^2$

So $y(x)=y_h+y_p$

$=c_1x+c_2x\ln x+x(\ln x)^2$

Method 2: $x = e^t \Rightarrow$ MUC.

By the chain rule, $x\frac{dy}{dx} = \frac{dy}{dt} = \tilde{y}'$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt} = \tilde{y}'' - \tilde{y}'$$

So our equation is now $\tilde{y}'' - \tilde{y}' - \tilde{y}' + \tilde{y} = 2e^t$

$$\tilde{y}'' - 2\tilde{y}' + \tilde{y} = 2e^t$$

$\tilde{y}'' - 2\tilde{y}' + \tilde{y} = 0$ Assume $\tilde{y}(t) = e^{rt}$

$\Rightarrow r^2 - 2r + 1 = (r-1)^2 = 0.$

$r_1 = 1 = r_2$

$\left.\begin{array}{l}\Rightarrow \tilde{y}_1(t) = e^t \\ \tilde{y}_2(t) = te^t\end{array}\right\} \tilde{y}_h = c_1e^t + c_2te^t.$

Note: $(D-1)(2e^t) = 0.$

Then $(D-1)^2\tilde{y} = 2e^t \Rightarrow (D-1)^3\tilde{y} = (D-1)(2e^t) = 0.$

Assume $\tilde{y}(t) = e^{rt}$

$\Rightarrow (r-1)^3 = 0$

$\Rightarrow \tilde{y}(t) = \underbrace{c_1e^t + c_2te^t}_{\tilde{y}_h} + \underbrace{c_3t^2e^t}_{\tilde{y}_p}.$

$\tilde{y}_p = At^2e^t$

$\tilde{y}_p' = 2Ate^t + At^2e^t$

$\tilde{y}_p'' = 2Ae^t + 4Ate^t + At^2e^t$

$\tilde{y}_p'' - 2\tilde{y}_p' + \tilde{y}_p = (A - 2A + A)t^2e^t + (4A - 2(2A))te^t + 2Ae^t$

$= 2Ae^t$

$= 2e^t$

$\Rightarrow A = 1$

$\Rightarrow \tilde{y}_p = t^2e^t$

So $y(t) = c_1e^t + c_2te^t + t^2e^t.$

Then $t = \ln x \Rightarrow y(x) = c_1e^{\ln x} + c_2e^{\ln x}(\ln x) + e^{\ln x}(\ln x)^2$

$= c_1x + c_2x(\ln x) + x(\ln x)^2.$

2. An object weighing 24 lbs stretches a spring 4 in. The object is displaced 3 in. above the equilibrium position and is released with no initial velocity. Assume that there is no damping force.

a. Set up, **but do not solve** the initial value problem describing this motion. **Use** $g = 32\text{ft}/\text{s}^2$ as the acceleration for gravity.

$W = mg = 24 \quad \Rightarrow \quad m = \frac{24}{32} = \frac{3}{4}$

$Ks = mg \quad \Rightarrow \quad \frac{4}{12}K = 24$

$K = 72.$

$\beta = 0$

$$\frac{3}{4}y'' + 72y = 0 \qquad \begin{cases} y(0) = -\frac{3}{12} = -\frac{1}{4} \\ y'(0) = 0 \end{cases}$$

b. Now assume that the object is acted on by an external force of $f(t) = 2\cos(\gamma t)$. For what value of γ does resonance occur?

$\frac{3}{4}y'' + 72y = 0 \qquad y(t) = e^{rt}$

$\Rightarrow \frac{3}{4}r^2 + 72 = 0$

$\frac{3}{4}r^2 = -72$

$r^2 = -96$

$r = \sqrt{-96} = 4\sqrt{6}\,i$

$\Rightarrow \omega = 4\sqrt{6} = \gamma$

3. Consider the differential equation $y''-4y=3\cos(x)$

a. Classify the differential equation by order, linearity, type of coefficients, and state whether or not the equation is homogeneous.

2nd order, linear, constant coefficients, nonhomogeneous.

b. What method(s) can you use to solve this equation?

y_h: use e^{rx} $\quad$ y_p: $\begin{cases} \text{MUC} \\ \text{VOP} \end{cases}$

c. Find a particular solution, y_p.

$y''-4y=0 \quad y(x)=e^{rx}$

$\Rightarrow r^2-4=0$

$(r-2)(r+2)=0$

$y_h(x)=c_1e^{2x}+c_2e^{-2x}$

Method 1: MUC.

$3\cos(x) \Rightarrow r=\pm i$

$\Rightarrow r^2+1=0$

$\Rightarrow D^2+1=0$

$(D^2-4)y=3\cos(x)$

$(D^2+1)(D^2-4)y=(D^2+1)(3\cos(x))=0$

$y(x)=e^{rx}$

$\Rightarrow (r^2+1)(r^2-4)=0$

$r=2,-2,\pm i$

$$y(x)=\underbrace{c_1e^{2x}+c_2e^{-2x}}_{y_h}+\underbrace{c_3\cos(x)+c_4\sin(x)}_{y_p}$$

$y_p=A\cos(x)+B\sin(x)$

$y_p'=-A\sin(x)+B\cos(x)$

$y_p''=-A\cos(x)-B\sin(x)$

$y_p''-4y_p=(-A-4A)\cos(x)+(-B-4B)\sin(x)$

$=-5A\cos(x)-5B\sin(x)$

$=3\cos(x)$

$\Rightarrow A=-\frac{3}{5} \quad B=0$

$y_p=-\frac{3}{5}\cos(x)$

$\Rightarrow y(x)=y_h+y_p=c_1e^{2x}+c_2e^{-2x}-\frac{3}{5}\cos(x).$

Method 2: VOP

$$W(y_1,y_2) = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -2-2=-4.$$

$$W_1 = \begin{vmatrix} 0 & e^{-2x} \\ 3\cos(x) & -2e^{-2x} \end{vmatrix} = -3e^{-2x}\cos(x).$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & 3\cos(x) \end{vmatrix} = 3e^{2x}\cos(x).$$

$$u_1' = \frac{W_1}{W} = \frac{3}{4}e^{-2x}\cos(x)$$

$$u_2' = \frac{W_2}{W} = -\frac{3}{4}e^{2x}\cos(x).$$

$$u_1(x) = \frac{3}{4}\int e^{-2x}\cos(x)\,dx = -\frac{3}{10}e^{-2x}\cos(x) + \frac{3}{20}e^{-2x}\sin(x)$$

$$u_2(x) = -\frac{3}{4}\int e^{2x}\cos(x)\,dx = -\frac{3}{10}e^{2x}\cos(x) - \frac{3}{20}e^{2x}\sin(x).$$

Now $y_p = u_1y_1 + u_2y_2$

$$\Rightarrow y_p = \left(-\frac{3}{10}e^{-2x}\cos(x) + \frac{3}{20}e^{-2x}\sin(x)\right)e^{2x}$$
$$+\left(-\frac{3}{10}e^{2x}\cos(x) - \frac{3}{20}e^{2x}\sin(x)\right)e^{-2x}$$
$$= -\frac{3}{10}\cos(x) + \frac{3}{20}\sin(x) - \frac{3}{10}\cos(x) - \frac{3}{20}\sin(x).$$
$$= -\frac{3}{5}\cos(x).$$

4. Find the inverse Laplace transform of the following:

a. $F(s) = \dfrac{s}{s^2+2s-3}$

$$= \frac{s}{(s+3)(s-1)}$$

$$= \frac{A}{s+3} + \frac{B}{s-1}$$

$$\Rightarrow s = A(s-1) + B(s+3)$$

$$s=-3 \quad -4A=-3 \Rightarrow A = \frac{3}{4}$$

$$s=1 \quad 4B=1 \quad \Rightarrow B = \frac{1}{4}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{\frac{3}{4}}{s+3} + \frac{\frac{1}{4}}{s-1}\right\} = \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= \frac{3}{4}e^{-3t} + \frac{1}{4}e^{t}$$

b. $G(s) = \dfrac{s}{s^2+4s+5}$

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4s+5}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+4s+\left(\frac{4}{2}\right)^2-\left(\frac{4}{2}\right)^2+5}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{(s+2)^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+2-2}{(s+2)^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\Big|_{s \to s+2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\Big|_{s \to s+2}\right\}$$

$$= e^{-2t}\cos(t) - 2e^{-2t}\sin(t)$$

5. Find the Laplace transform to solve the initial value problem $y' - y = 2e^{-3t}$ subject to $y(0) = 0$.

$$\mathcal{L}\{y' - y\} = \mathcal{L}\{2e^{-3t}\}$$

$$\mathcal{L}\{y'\} - \mathcal{L}\{y\} = 2\mathcal{L}\{e^{-3t}\}$$

$$sY(s) - y(0) - Y(s) = \frac{2}{s+3}$$

$$(s-1)Y(s) = \frac{2}{s+3}$$

$$Y(s) = \frac{2}{(s-1)(s+3)}$$

$$= \frac{A}{(s-1)} + \frac{B}{s+3}$$

$$\Rightarrow 2 = A(s+3) + B(s-1)$$

$$s = 1, \quad 4A = 2 \Rightarrow A = \frac{1}{2}$$

$$s = -3, \quad -4B = 2 \Rightarrow B = -\frac{1}{2}$$

$$\Rightarrow Y(s) = \frac{\frac{1}{2}}{s-1} - \frac{\frac{1}{2}}{s+3}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{\frac{1}{2}}{s-1} - \frac{\frac{1}{2}}{s+3}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{1}{2}e^{t} - \frac{1}{2}e^{-3t}$$

Bonus (15 points): Use the Laplace transform to solve the initial value problem $y'+y=1-2\mathcal{U}(t-1)$ subject to $y(0)=0$. Graph your solution.

$$\mathcal{L}\{y'+y\}=\mathcal{L}\{1-2\,\mathcal{U}(t-1)\}$$
$$\mathcal{L}\{y'\}+\mathcal{L}\{y\}=\mathcal{L}\{1\}-2\mathcal{L}\{\mathcal{U}(t-1)\}$$
$$sY(s)-\underset{\overset{\parallel}{0}}{y(0)}+Y(s)=\frac{1}{s}-2\frac{e^{-s}}{s}.$$

$$(s+1)Y(s)=\frac{1}{s}-\frac{2e^{-s}}{s}.$$
$$Y(s)=\frac{1}{s(s+1)}-\frac{2e^{-s}}{s(s+1)}.$$

$$\text{Let } F(s)=\frac{1}{s(s+1)}=\frac{A}{s}+\frac{B}{s+1}=\frac{1}{s}-\frac{1}{s+1}$$
$$\Rightarrow 1=A(s+1)+Bs$$
$$s=0 \Rightarrow A=1$$
$$s=-1 \Rightarrow B=-1.$$
$$\text{Then } f(t)=\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}=1-e^{-t}$$

$$Y(s)=F(s)-2e^{-s}F(s)$$
$$\text{Then } y(t)=\mathcal{L}^{-1}\{F(s)\}-2\mathcal{L}^{-1}\{e^{-s}F(s)\}$$
$$=1-e^{-t}-2(1-e^{-(t-1)})\mathcal{U}(t-1)$$
$$=\begin{cases}1-e^{-t} & 0\le t<1\\ -1-e^{-t}+2e^{-(t-1)} & t\ge 1.\end{cases}$$