MTH 204

Fall 2005

Exam 4

Name: Key

Section: A/C

Math 204

Sections A and C Exams

Exam 4 Sec. A C Fall 2005

Read the directions carefully.

Write <u>neatly</u> in pencil and **<u>show all your work</u>**

<u>(NO WORK, NO CREDIT)</u>.

Each question is worth 20 points

Please do not share calculators during the test.

The last page contains your Laplace tables.

If you have trouble during the test, feel free to ask me for help.

Score:________

1. Find the inverse Laplace transform of the following

a. $F(s) = \dfrac{5}{s^2+49}$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = 5\mathcal{L}^{-1}\left\{\left(\frac{7}{7}\right)\left(\frac{1}{s^2+49}\right)\right\} = \frac{5}{7}\sin(7t)$$

b. $G(s) = \dfrac{s}{s^2+2s-3}$

$$\frac{s}{s^2+2s-3} = \frac{s}{(s+3)(s-1)} = \frac{A}{s+3} + \frac{B}{s-1} \Rightarrow s = A(s-1) + B(s+3)$$

$$s=-3 \quad -3 = -4A \Rightarrow A = \frac{3}{4}$$

$$s = 1 \quad 1 = 4B \Rightarrow B = \frac{1}{4}$$

$$g(t) = \mathcal{L}^{-1}\left\{\frac{\frac{3}{4}}{s+3} + \frac{\frac{1}{4}}{s-1}\right\} = \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= \frac{3}{4}e^{-3t} + \frac{1}{4}e^{t}$$

c. $H(s) = \dfrac{s}{s^2+4s+5}$

$$\frac{s}{s^2+4s+5} = \frac{s}{s^2+4s+\left(\frac{4}{2}\right)^2-\left(\frac{4}{2}\right)^2+5} = \frac{s}{(s+2)^2+1}$$

$$h(t) = \mathcal{L}^{-1}\left\{\frac{s}{(s+2)^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{s+2-2}{(s+2)^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}_{s\to s+2} - 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}_{s\to s+2} = e^{-2t}\cos(t) - 2e^{-2t}\sin(t)$$

2. Solve the initial value problem $y''+3y'+2y = f(t)$ subject to $\begin{cases} y(0)=0 \\ y'(0)=0 \end{cases}$, with

$$f(t)=\begin{cases} 1, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}.$$

$f(t) = 1(\mathcal{U}(t-0) - \mathcal{U}(t-10)) - 0\,\mathcal{U}(t-10) = 1 - \mathcal{U}(t-10).$

$y''+3y'+2y = 1-\mathcal{U}(t-10)$

$\mathcal{L}\{y''+3y'+2y\} = \mathcal{L}\{1-\mathcal{U}(t-10)\}$

$s^2Y(s) - s\overset{\to 0}{y(0)} - \overset{\to 0}{y'(0)} + 3(sY(s) - \overset{\to 0}{y(0)}) + 2Y(s) = \frac{1}{s} - \frac{e^{-10s}}{s}$

$(s^2+3s+2)Y(s) = \frac{1}{s} - \frac{e^{-10s}}{s}$

$$Y(s) = \underbrace{\frac{1}{s(s+1)(s+2)}}_{F(s)} - \underbrace{\frac{e^{-10s}}{s(s+1)(s+2)}}_{e^{-10s}F(s)}$$

$F(s) = \frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} = \frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}$

$1 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$

$s=0 \quad 1 = 2A \Rightarrow A = \frac{1}{2}$

$s=-1 \quad 1 = -B \Rightarrow B = -1$

$s=-2 \quad 1 = 2C \Rightarrow C = \frac{1}{2}$

$f(t) = \mathcal{L}^{-1}\left\{\frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$

$= \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$

$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)(s+2)} - \frac{e^{-10s}}{s(s+1)(s+2)}\right\}$

$= \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} - \left(\frac{1}{2} - e^{-(t-10)} + \frac{1}{2}e^{-2(t-10)}\right)\mathcal{U}(t-10)$

3. **Solve the initial problem** $y''+2y'=\delta(t-1)$ **subject to** $\begin{cases} y(0)=0 \\ y'(0)=1 \end{cases}$.

$\mathcal{L}\{y''+2y'\} = \mathcal{L}\{\delta(t-1)\}$

$s^2Y(s) - s\overset{\to 0}{y(0)} - \overset{\to 1}{y'(0)} + 2(sY(s) - \overset{\to 0}{y(0)}) = e^{-s}$

$(s^2+2s)Y(s) - 1 = e^{-s}$

$$Y(s) = \underbrace{\frac{1}{s(s+2)}}_{F(s)} + \underbrace{\frac{e^{-s}}{s(s+2)}}_{e^{-s}F(s)}$$

$$F(s) = \frac{1}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} = \frac{\frac{1}{2}}{s} - \frac{\frac{1}{2}}{s+2}$$

$1 = A(s+2) + Bs$

$s=0 \quad 1 = 2A \Rightarrow A = \frac{1}{2}$

$s=-2 \quad 1 = -2B \Rightarrow B = -\frac{1}{2}$.

$$f(t) = \mathcal{L}^{-1}\left\{\frac{\frac{1}{2}}{s} - \frac{\frac{1}{2}}{s+2}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = \frac{1}{2} - \frac{1}{2}e^{-2t}$$

So $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$= \mathcal{L}^{-1}\{F(s)\} + \mathcal{L}^{-1}\{e^{-s}F(s)\}$

$= \frac{1}{2} - \frac{1}{2}e^{-2t} + \left(\frac{1}{2} - \frac{1}{2}e^{-2(t-10)}\right)u(t-10)$

4. Use the Laplace transform to solve the integral equation $f(t)+\int_0^t (t-\tau)f(\tau)d\tau = t$ for $f(t)$.

$$\mathcal{L}\{f(t)+\int_0^t f(\tau)(t-\tau)d\tau\} = \mathcal{L}\{t\}$$

$$\mathcal{L}\{f(t)\} + \mathcal{L}\{\int_0^t f(\tau)(t-\tau)d\tau\} = \mathcal{L}\{t\}$$

$$F(s) + \frac{F(s)}{s^2} = \frac{1}{s^2}$$

$$\left(\frac{s^2+1}{s^2}\right)F(s) = \frac{1}{s^2}$$

$$F(s) = \frac{1}{s^2+1}$$

So $f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin(t)$.

5. Consider the vectors $\vec{X}_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix} e^{8t}$ and $\vec{X}_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix} e^{-10t}$.

a. Determine whether $\vec{X}_1$ and $\vec{X}_2$ are linearly independent on $(-\infty, \infty)$.

b. Do $\vec{X}_1$ and $\vec{X}_2$ form a fundamental set of solutions for $\vec{X}' = \begin{pmatrix} 10 & -5 \\ 8 & -12 \end{pmatrix} \vec{X}$ on $(-\infty, \infty)$?

State why or why not.

a. $W(\vec{x}_1, \vec{x}_2) = \begin{vmatrix} 5e^{8t} & 2e^{-10t} \\ 2e^{8t} & 4e^{-10t} \end{vmatrix} = 20e^{-2t} - 4e^{-2t} = 16e^{-2t} \neq 0$

$\Rightarrow$ Linearly independent on $(-\infty, \infty)$.

b. $\vec{x}_1' = 8 \begin{pmatrix} 5 \\ 2 \end{pmatrix} e^{8t} = \begin{pmatrix} 40 \\ 16 \end{pmatrix} e^{8t}$

$\begin{bmatrix} 10 & -5 \\ 8 & -12 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^{8t} = \begin{bmatrix} 50-10 \\ 40-24 \end{bmatrix} e^{8t} = \begin{bmatrix} 40 \\ 16 \end{bmatrix} e^{8t}$

So $\vec{x}' = \begin{bmatrix} 10 & -5 \\ 8 & -12 \end{bmatrix} \vec{x}_1 \Rightarrow \vec{x}_1$ is a solution.

$\vec{x}_2' = -10 \begin{bmatrix} 2 \\ 4 \end{bmatrix} e^{-10t} = \begin{bmatrix} -20 \\ -40 \end{bmatrix} e^{-10t}$

$\begin{bmatrix} 10 & -5 \\ 8 & -12 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} e^{-10t} = \begin{bmatrix} 20-20 \\ 16-48 \end{bmatrix} e^{-10t} = \begin{bmatrix} 0 \\ -32 \end{bmatrix} e^{-10t}$

$\vec{x}_2' \neq \begin{bmatrix} 10 & -5 \\ 8 & -12 \end{bmatrix} \vec{x}_2 \Rightarrow \vec{x}_2$ is not a solution.

Since $\vec{x}_2$ is not a solution, $\vec{x}_1, \vec{x}_2$ do not form a FSS.

Bonus. Use the Laplace transform to solve the following system of initial value problems

$$\frac{d^2x}{dt^2}+\frac{d^2y}{dt^2}=t^2 \qquad \text{subject to} \quad \begin{cases} x(0)=8 & y(0)=0 \\ x'(0)=0 & y'(0)=0 \end{cases}$$
$$\frac{d^2x}{dt^2}-\frac{d^2y}{dt^2}=4t$$

$$\begin{cases} \mathcal{L}\{x''+y''\}=\mathcal{L}\{t^2\} \\ \mathcal{L}\{x''-y''\}=\mathcal{L}\{4t\} \end{cases}$$

$$\begin{cases} s^2X(s)-s\overset{\to 8}{x(0)}-\overset{\to 0}{x'(0)}+s^2Y(s)-s\overset{\to 0}{y(0)}-\overset{\to 0}{y'(0)}=\frac{2}{s^3} \\ s^2X(s)-s\overset{\to 8}{x(0)}-\overset{\to 0}{x'(0)}-(s^2Y(s)-s\overset{\to 0}{y(0)}-\overset{\to 0}{y'(0)})=\frac{4}{s^2} \end{cases}$$

$$(+)\begin{cases} s^2X(s)+s^2Y(s)=\frac{2}{s^3}+8s \\ s^2X(s)-s^2Y(s)=\frac{4}{s^2}+8s \end{cases}$$

$$2s^2X(s)=\frac{2}{s^3}+\frac{4}{s^2}+16s$$

$$X(s)=\frac{1}{s^5}+\frac{2}{s^4}+\frac{8}{s}.$$

$$x(t)=\mathcal{L}^{-1}\left\{\frac{1}{s^5}+\frac{2}{s^4}+\frac{8}{s}\right\}=\mathcal{L}^{-1}\left\{\left(\frac{4!}{4!}\right)\left(\frac{1}{s^5}\right)\right\}+\mathcal{L}^{-1}\left\{\left(\frac{3}{3}\right)\left(\frac{2}{s^4}\right)\right\}+8\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$
$$=\frac{1}{4!}t^4+\frac{1}{3}t^3+8.$$

$$(-)\begin{cases} s^2X(s)+s^2Y(s)=\frac{2}{s^3}+8s \\ s^2X(s)-s^2Y(s)=\frac{4}{s^2}+8s \end{cases}$$

$$2s^2Y(s)=\frac{2}{s^3}+8s-\frac{4}{s^2}-8s=\frac{2}{s^3}-\frac{4}{s^2}$$

$$Y(s)=\frac{1}{s^5}-\frac{2}{s^4}$$

$$y(t)=\mathcal{L}^{-1}\left\{\frac{1}{s^5}-\frac{2}{s^4}\right\}=\mathcal{L}^{-1}\left\{\left(\frac{4!}{4!}\right)\left(\frac{1}{s^5}\right)\right\}-\mathcal{L}^{-1}\left\{\left(\frac{3}{3}\right)\left(\frac{2}{s^4}\right)\right\}=\frac{1}{4!}t^4-\frac{1}{3}t^3$$

So $\begin{cases} x(t)=\frac{1}{4!}t^4+\frac{1}{3}t^3+8 \\ y(t)=\frac{1}{4!}t^4-\frac{1}{3}t^3 \end{cases}$

TABLE OF LAPLACE TRANSFORMS

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
1. 1	$\dfrac{1}{s}$
2. t	$\dfrac{1}{s^2}$
3. t^n	$\dfrac{n!}{s^{n+1}}$, n a positive integer
4. $t^{-1/2}$	$\sqrt{\dfrac{\pi}{s}}$
5. $t^{1/2}$	$\dfrac{\sqrt{\pi}}{2s^{3/2}}$
6. t^α	$\dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \alpha > -1$
7. $\sin kt$	$\dfrac{k}{s^2+k^2}$
8. $\cos kt$	$\dfrac{s}{s^2+k^2}$
9. $\sin^2 kt$	$\dfrac{2k^2}{s(s^2+4k^2)}$
10. $\cos^2 kt$	$\dfrac{s^2+2k^2}{s(s^2+4k^2)}$
11. e^{at}	$\dfrac{1}{s-a}$
12. $\sinh kt$	$\dfrac{k}{s^2-k^2}$
13. $\cosh kt$	$\dfrac{s}{s^2-k^2}$
14. $\sinh^2 kt$	$\dfrac{2k^2}{s(s^2-4k^2)}$
15. $\cosh^2 kt$	$\dfrac{s^2-2k^2}{s(s^2-4k^2)}$
16. te^{at}	$\dfrac{1}{(s-a)^2}$
17. $t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$, n a positive integer
18. $e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$
19. $e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$
20. $e^{at}\sinh kt$	$\dfrac{k}{(s-a)^2-k^2}$
21. $e^{at}\cosh kt$	$\dfrac{s-a}{(s-a)^2-k^2}$
22. $t\sin kt$	$\dfrac{2ks}{(s^2+k^2)^2}$
23. $t\cos kt$	$\dfrac{s^2-k^2}{(s^2+k^2)^2}$
24. $\sin kt + kt\cos kt$	$\dfrac{2ks^2}{(s^2+k^2)^2}$
25. $\sin kt - kt\cos kt$	$\dfrac{2k^3}{(s^2+k^2)^2}$
26. $t\sinh kt$	$\dfrac{2ks}{(s^2-k^2)^2}$
27. $t\cosh kt$	$\dfrac{s^2+k^2}{(s^2-k^2)^2}$
28. $\dfrac{e^{at}-e^{bt}}{a-b}$	$\dfrac{1}{(s-a)(s-b)}$
29. $\dfrac{ae^{at}-be^{bt}}{a-b}$	$\dfrac{s}{(s-a)(s-b)}$
30. $1-\cos kt$	$\dfrac{k^2}{s(s^2+k^2)}$
31. $kt-\sin kt$	$\dfrac{k^3}{s^2(s^2+k^2)}$
32. $\dfrac{a\sin bt - b\sin at}{ab(a^2-b^2)}$	$\dfrac{1}{(s^2+a^2)(s^2+b^2)}$
33. $\dfrac{\cos bt - \cos at}{a^2-b^2}$	$\dfrac{s}{(s^2+a^2)(s^2+b^2)}$
34. $\sin kt\sinh kt$	$\dfrac{2k^2 s}{s^4+4k^4}$

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
35. $\sin kt \cosh kt$	$\dfrac{k(s^2 + 2k^2)}{s^4 + 4k^4}$
36. $\cos kt \sinh kt$	$\dfrac{k(s^2 - 2k^2)}{s^4 + 4k^4}$
37. $\cos kt \cosh kt$	$\dfrac{s^3}{s^4 + 4k^4}$
38. $J_0(kt)$	$\dfrac{1}{\sqrt{s^2 + k^2}}$
39. $\dfrac{e^{bt} - e^{at}}{t}$	$\ln \dfrac{s - a}{s - b}$
40. $\dfrac{2(1 - \cos kt)}{t}$	$\ln \dfrac{s^2 + k^2}{s^2}$
41. $\dfrac{2(1 - \cosh kt)}{t}$	$\ln \dfrac{s^2 - k^2}{s^2}$
42. $\dfrac{\sin at}{t}$	$\arctan \left(\dfrac{a}{s}\right)$
43. $\dfrac{\sin at \cos bt}{t}$	$\dfrac{1}{2} \arctan \dfrac{a + b}{s} + \dfrac{1}{2} \arctan \dfrac{a - b}{s}$
44. $\delta(t)$	1
45. $\delta(t - t_0)$	e^{-st_0}
46. $e^{at}f(t)$	$F(s - a)$
47. $f(t - a)\mathcal{U}(t - a)$	$e^{-as}F(s)$
48. $\mathcal{U}(t - a)$	$\dfrac{e^{-as}}{s}$
49. $f^{(n)}(t)$	$s^nF(s) - s^{(n-1)}f(0) - \cdots - f^{(n-1)}(0)$
50. $t^nf(t)$	$(-1)^n \dfrac{d^n}{ds^n} F(s)$
51. $\int_0^t f(\tau)g(t - \tau)\, d\tau$	$F(s)G(s)$