MTH 204
Fall 2006
Exam 4

Name: Key

Section: B

Read the directions carefully.
Write neatly in pencil and show all your work
(you will only get credit for what you put on paper).
Please do not share calculators during the test.
Each question is worth 25 points
You must calculate all eigenvalues and eigenvectors
by hand. The last page contains your Laplace tables.
If you have trouble during the test, feel free to ask me for help.

1. Use the Laplace transform to solve the integral equation $f(t) = t - \int_0^t (t - \tau) f(\tau) d\tau$.

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t\} - \mathcal{L}\left\{\int_0^t (t-\tau) f(\tau) d\tau\right\}$$

$$F(s) = \frac{1}{s^2} - \mathcal{L}\{f(t) * t\}$$

$g(t-\tau) = t-\tau$
$\Rightarrow g(t) = t$

$$= \frac{1}{s^2} - \mathcal{L}\{f(t)\}\mathcal{L}\{t\}$$

$$= \frac{1}{s^2} - \frac{F(s)}{s^2}$$

$$\Rightarrow F(s) + \frac{F(s)}{s^2} = \left(\frac{s^2+1}{s^2}\right) F(s) = \frac{1}{s^2}$$

$$F(s) = \frac{1}{s^2}\left(\frac{s^2}{s^2+1}\right) = \frac{1}{s^2+1}$$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin(t)$$

2. Solve the following initial value problem $\bar{x}' = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \bar{x}$, $\bar{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Method 1: $0 = \det(A - \lambda I)$

$$= \lambda^2 - \text{Tr}A\lambda + \det A$$
$$= \lambda^2 - 10\lambda + 25 - 9$$
$$= \lambda^2 - 10\lambda + 16$$
$$= (\lambda - 8)(\lambda - 2)$$

$\Rightarrow \lambda_1 = 8, \lambda_2 = 2$

For $\lambda_1 = 8$, $(A - \lambda_1 I)\vec{K}_1 = \vec{0}$

$$\left[\begin{array}{cc|c} 5-8 & 3 & 0 \\ 3 & 5-8 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} -3 & 3 & 0 \\ \cancel{3} & \cancel{-3} & \cancel{0} \end{array}\right]$$

$\Rightarrow -k_1 + k_2 = 0$

$\Rightarrow k_1 = k_2$

$\hookrightarrow FV = 1$

$\Rightarrow \vec{K}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Rightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{8t}$

For $\lambda_2 = 2$, $(A - \lambda_2 I)\vec{K}_2 = \vec{0}$

$$\left[\begin{array}{cc|c} 5-2 & 3 & 0 \\ 3 & 5-2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 3 & 3 & 0 \end{array}\right]$$

$\Rightarrow k_1 + k_2 = 0$

$\Rightarrow k_2 = -k_1$

$\hookrightarrow FV = 1$

$\Rightarrow \vec{K}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\Rightarrow \vec{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$

GS: $\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{8t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$

$\vec{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\Rightarrow$

$$c_1 + c_2 = 1$$
$$c_1 - c_2 = -1$$
$$\overline{2c_1 = 0} \Rightarrow c_1 = 0$$
$$\Rightarrow c_2 = 1$$

So $\vec{x}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$

Method 2: $x' = 5x + 3y$ $\quad x(0) = 1$
$y' = 3x + 5y$ $\quad y(0) = -1$

$$\begin{cases} \mathcal{L}\{x' - 5x - 3y\} = 0 \\ \mathcal{L}\{-3x + y' - 5y\} = 0 \end{cases}$$

$$\begin{cases} sX(s) - x(0) - 5X(s) - 3Y(s) = 0 \\ -3X(s) + sY(s) - y(0) - 5Y(s) = 0 \end{cases}$$

$$\begin{array}{r} 3\left[(s-5)X(s) - 3Y(s) = 1 \right] \\ +(s-5)\left[-3X(s) + (s-5)Y(s) = -1 \right] \\ \hline \end{array}$$

$$(-9 + (s-5)^2)Y(s) = 3 - (s-5)$$

$$(s-8)(s-2)Y(s) = -(s-8)$$

$$Y(s) = \frac{-1}{s-2}$$

$$\Rightarrow y(t) = -e^{2t}$$

$$\begin{array}{r} (s-5)\left[(s-5)X(s) - 3Y(s) = 1 \right] \\ +3\left[-3X(s) + (s-5)Y(s) = -1 \right] \\ \hline \end{array}$$

$$((s-5)^2 - 9)X(s) = s - 5 - 3$$

$$(s-2)(s-8)X(s) = s - 8$$

$$X(s) = \frac{1}{s-2}$$

$$\Rightarrow x(t) = e^{2t}$$

Then $\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$$

3. Find the general solution to the following homogeneous linear system $\vec{x}' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \vec{x}$.

$$0 = \det(A - \lambda I) = \lambda^2 - \mathrm{Tr}A\,\lambda + \det A = \lambda^2 - 2\lambda - 3 + 4 = (\lambda - 1)^2$$

$$\Rightarrow \lambda = 1, 1$$

$$(A - \lambda I)\vec{K} = \vec{0}$$

$$\left[\begin{array}{cc|c} 3-1 & -4 & 0 \\ 1 & -1-1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} \cancel{2} & \cancel{-4} & \cancel{0} \\ 1 & -2 & 0 \end{array}\right]$$

$$\Rightarrow k_1 - 2k_2 = 0$$

$$\Rightarrow k_1 = 2k_2$$

$\hookrightarrow$ FV=1

$$\Rightarrow \vec{K} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t$$

$$(A - \lambda I)\vec{P} = \vec{K}$$

$$\left[\begin{array}{cc|c} 3-1 & -4 & 2 \\ 1 & -1-1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} \cancel{2} & \cancel{-4} & \cancel{2} \\ 1 & -2 & 1 \end{array}\right]$$

$$\Rightarrow p_1 - 2p_2 = 1$$

$$\Rightarrow p_1 = 1 + 2p_2$$

$$\Rightarrow \vec{P} = \begin{bmatrix} 1 + 2p_2 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + p_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $p_2 = 0$

$$\Rightarrow \vec{P} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \vec{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t$$

GS: $\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2$

$$= c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t \right)$$

4. Find the general solution to the following homogeneous linear system $\vec{x}' = \begin{bmatrix} -1 & 1 \\ -5 & 3 \end{bmatrix} \vec{x}$.

$$0 = \det(A - \lambda I) = \lambda^2 - \mathrm{Tr}A\,\lambda + \det A = \lambda^2 - 2\lambda - 3 + 5 = \lambda^2 - 2\lambda + 2$$

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{4 - 4(2)}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

Let $\lambda = 1 + i \Rightarrow (A - \lambda I)\vec{K} = \vec{0}$

$$\left[\begin{array}{cc|c} -1-(1+i) & 1 & 0 \\ -5 & 3-(1+i) & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} -(2+i) & 1 & 0 \\ \cancel{-5} & \cancel{2-i} & \cancel{0} \end{array}\right]$$

$$\Rightarrow -(2+i)k_1 + k_2 = 0$$

$$\Rightarrow k_2 = (2+i)\underbrace{k_1}_{FV=1}$$

$$\Rightarrow \vec{K} = \begin{bmatrix} 1 \\ 2+i \end{bmatrix}$$

$$z = \vec{K}e^{\lambda t} = \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)e^{(1+i)t} = \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)e^t(\cos(t) + i\sin(t))$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix}e^t\cos(t) + i\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^t\sin(t) + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}e^t\cos(t) - \begin{bmatrix} 0 \\ 1 \end{bmatrix}e^t\sin(t)$$

$$\vec{x}_1 = \mathrm{Re}\,\vec{z} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}e^t\cos(t) - \begin{bmatrix} 0 \\ 1 \end{bmatrix}e^t\sin(t) = \begin{bmatrix} \cos(t) \\ 2\cos(t) - \sin(t) \end{bmatrix}e^t$$

$$\vec{x}_2 = \mathrm{Im}\,\vec{z} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}e^t\sin(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}e^t\cos(t) = \begin{bmatrix} \sin(t) \\ 2\sin(t) + \cos(t) \end{bmatrix}e^t$$

GS: $\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2$

$$= c_1\begin{bmatrix} \cos(t) \\ 2\cos(t) - \sin(t) \end{bmatrix}e^t + c_2\begin{bmatrix} \sin(t) \\ 2\sin(t) + \cos(t) \end{bmatrix}e^t$$

Bonus (15 points): Use the Laplace transform to solve the following system of IVPs

$$\frac{d^2x}{dt^2}+\frac{d^2y}{dt^2}=t^2 \qquad \frac{d^2x}{dt^2}-\frac{d^2y}{dt^2}=3t \qquad \text{subject to } \begin{cases} x(0)=-1 & y(0)=0 \\ x'(0)=0 & y'(0)=0 \end{cases}$$

$$\begin{cases} \mathcal{L}\{x''+y''\}=\mathcal{L}\{t^2\} \\ \mathcal{L}\{x''-y''\}=3\mathcal{L}\{t\} \end{cases}$$

$$\begin{cases} s^2X(s)-sX(0)-x'(0)+s^2Y(s)-sy(0)-y'(0)=\frac{2}{s^3} \\ s^2X(s)-sX(0)-x'(0)-(s^2Y(s)-sy(0)-y'(0))=\frac{3}{s^2} \end{cases}$$

$$s^2X(s)+s^2Y(s)=\frac{2}{s^3}-s$$

$$+\ s^2X(s)-s^2Y(s)=\frac{3}{s^2}-s$$

$$2s^2X(s)=\frac{2}{s^3}+\frac{3}{s^2}-2s$$

$$X(s)=\frac{1}{s^5}+\frac{3/2}{s^4}-\frac{1}{s}$$

$$x(t)=\frac{1}{4!}\mathcal{L}^{-1}\left\{\frac{4!}{s^{4+1}}\right\}+\frac{3}{2(3!)}\mathcal{L}^{-1}\left\{\frac{3!}{s^{3+1}}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}=\frac{1}{24}t^4+\frac{1}{4}t^3-1$$

$$s^2X(s)+s^2Y(s)=\frac{2}{s^3}-s$$

$$-\ s^2X(s)-s^2Y(s)=\frac{3}{s^2}-s$$

$$2s^2Y(s)=\frac{2}{s^3}-\frac{3}{s^2}$$

$$Y(s)=\frac{1}{s^5}-\frac{3/2}{s^4} \quad \Rightarrow \quad y(t)=\frac{1}{24}t^4-\frac{1}{4}t^3$$

$$\text{So } \begin{cases} x(t)=\frac{1}{24}t^4+\frac{1}{4}t^3-1 \\ y(t)=\frac{1}{24}t^4-\frac{1}{4}t^3 \end{cases}$$