MTH 204
Spring 2007
Exam 4

Name: Key

Section: B or C (circle one)

Read the directions carefully.
Write neatly in pencil and show all your work
(you will only get credit for what you put on paper).
Please do not share calculators during the test.
Each question is worth 20 points
DO NOT USE Decimals on any intermediate step.
The last page contains your Laplace tables.
If you have trouble during the test, feel free to ask me for help.

1. Use the Laplace transform to solve the integral equation $f(t) = 3e^t - \int_0^t e^{t-\tau} f(\tau)d\tau$ for f(t).

$g(t-\tau) = e^{t-\tau} \Rightarrow g(t) = e^t$

$f(t) = 3e^t - (e^t * f(t))$

$\mathcal{L}\{f(t)\} = 3\mathcal{L}\{e^t\} - \mathcal{L}\{e^t\}\mathcal{L}\{f(t)\}$

$F(s) = \frac{3}{s-1} - \frac{F(s)}{s-1}$

$\left(1 + \frac{1}{s-1}\right)F(s) = \frac{3}{s-1}$

$\left(\frac{s}{s-1}\right)F(s) = \frac{3}{s-1}$

$F(s) = \frac{3}{s-1}\left(\frac{s-1}{s}\right) = \frac{3}{s}$

$f(t) = \mathcal{L}^{-1}\{F(s)\} = 3$

2. Consider the IVP $y''+y=\delta(t-\frac{\pi}{2})+\delta(t-\frac{3\pi}{2})$ subject to $y(0)=0, y'(0)=0$.

a. Solve for y(t).

$$\mathcal{L}\{y''\}+\mathcal{L}\{y\}=\mathcal{L}\{\delta(t-\pi/2)\}+\mathcal{L}\{\delta(t-\tfrac{3\pi}{2})\}$$

$$s^2Y(s)-s\underset{0}{\underset{\parallel}{y(0)}}-\underset{0}{\underset{\parallel}{y'(0)}}+Y(s)=e^{-\frac{\pi}{2}s}+e^{-\frac{3\pi}{2}s}$$

$$(s^2+1)Y(s)=e^{-\pi/2\, s}+e^{-\frac{3\pi}{2}s}$$

$$Y(s)=\frac{e^{-\pi/2\, s}}{s^2+1}+\frac{e^{-\frac{3\pi}{2}s}}{s^2+1}=e^{-\frac{\pi}{2}s}F(s)+e^{-\frac{3\pi}{2}s}F(s)$$

$$F(s)=\frac{1}{s^2+1}\Rightarrow f(t)=\sin(t)$$

Recall $\mathcal{L}^{-1}\{e^{-as}F(s)\}=f(t-a)\mathcal{U}(t-a)$

$$\begin{aligned}y(t)&=\mathcal{L}^{-1}\{Y(s)\}\\&=\mathcal{L}^{-1}\{e^{-\frac{\pi}{2}s}F(s)\}+\mathcal{L}^{-1}\{e^{-\frac{3\pi}{2}s}F(s)\}\\&=\sin(t-\tfrac{\pi}{2})\mathcal{U}(t-\tfrac{\pi}{2})+\sin(t-\tfrac{3\pi}{2})\mathcal{U}(t-\tfrac{3\pi}{2})\\&=-\cos(t)\mathcal{U}(t-\tfrac{\pi}{2})+\cos(t)\mathcal{U}(t-\tfrac{3\pi}{2})\\&=\begin{cases}0, & 0\le t<\pi/2\\-\cos(t), & \frac{\pi}{2}\le t<\frac{3\pi}{2}\\-\cos(t)+\cos(t)=0, & t\ge\frac{3\pi}{2}\end{cases}\end{aligned}$$

b. Evaluate the following.

i. $y(\pi)=$ $\underline{-\cos(\pi)=1}$

ii. $y(\frac{7\pi}{2})=$ $\underline{0}$

3. Solve the following initial value problem $\vec{x}' = \begin{bmatrix} -2 & 1 \\ -5 & 4 \end{bmatrix}\vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

$$\begin{aligned} 0 = \det(A-\lambda I) &= \lambda^2 - \mathrm{Tr}A\,\lambda + \det A \\ &= \lambda^2 - 2\lambda + (-8) - (-5) \\ &= \lambda^2 - 2\lambda - 3 \\ &= (\lambda-3)(\lambda+1) \end{aligned}$$

$$\Rightarrow \lambda_1 = 3, \quad \lambda_2 = -1$$

For $\lambda_1 = 3$

$(A-\lambda_1 I)\vec{K}_1 = \vec{0}$

$$\begin{bmatrix} -2-3 & 1 \\ -5 & 4-3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -5 & 1 & 0 \\ \cancel{-5} & \cancel{1} & \cancel{0} \end{array}\right] \quad R_2 = R_1$$

$-5u_1 + u_2 = 0$

$u_2 = 5u_1 \hookrightarrow FV = 1$

$$\vec{K}_1 = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$\Rightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ 5 \end{bmatrix} e^{3t}$$

For $\lambda_2 = -1$

$(A-\lambda_2 I)\vec{K}_2 = \vec{0}$

$$\begin{bmatrix} -2-(-1) & 1 \\ -5 & 4-(-1) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ \cancel{-5} & \cancel{5} & \cancel{0} \end{array}\right] \quad R_2 = 5R_1$$

$-v_1 + v_2 = 0$

$v_2 = v_1 \hookrightarrow FV = 1$

$$\vec{K}_2 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

$$\Rightarrow \vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

$$\vec{x}(0) = c_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{aligned} c_1 + c_2 &= 1 \\ -(5c_1 + c_2 &= 3) \\ -4c_1 &= -2 \quad \Rightarrow c_1 = c_2 = 1/2 \end{aligned}$$

$$\Rightarrow \vec{x}(t) = \frac{1}{2}\begin{bmatrix} 1 \\ 5 \end{bmatrix} e^{3t} + \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

4. Find the general solution to the following homogeneous linear system $\bar{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \bar{x}$.

$$0 = \det(A - \lambda I) = \lambda^2 - \mathrm{Tr}A\lambda + \det A$$
$$= \lambda^2 - 4\lambda + 3 - (-1)$$
$$= \lambda^2 - 4\lambda + 4$$
$$= (\lambda - 2)^2$$
$$\Rightarrow \lambda = 2, 2$$

$$(A - \lambda I)\vec{K} = \vec{0}$$
$$\begin{bmatrix} 1-2 & -1 \\ 1 & 3-2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\left[\begin{array}{cc|c} \cancel{-1} & \cancel{-1} & \cancel{0} \\ 1 & 1 & 0 \end{array}\right] \quad R_1 = -R_2$$
$$u_1 + u_2 = 0$$
$$u_2 = -\underbrace{u_1}_{FV=1}$$
$$\vec{K} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$\Rightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$$

$$(A - \lambda I)\vec{P} = \vec{K}$$
$$\begin{bmatrix} 1-2 & -1 \\ 1 & 3-2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$\left[\begin{array}{cc|c} \cancel{-1} & \cancel{-1} & \cancel{1} \\ 1 & 1 & -1 \end{array}\right] \quad R_1 = -R_2$$
$$p_1 + p_2 = -1$$
$$p_2 = -1 - \underbrace{p_1}_{FV=0}$$
$$\vec{P} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$
$$\Rightarrow \vec{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t}$$

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{2t} \right)$$

5. Find the general solution to the following homogeneous linear system $\vec{x}' = \begin{bmatrix} 6 & -1 \\ 5 & 2 \end{bmatrix} \vec{x}$.

$$0 = \det(A - \lambda I) = \lambda^2 - \operatorname{Tr}A\,\lambda + \det A$$

$$= \lambda^2 - 8\lambda + 12 - (-5)$$

$$= \lambda^2 - 8\lambda + 17$$

$$\lambda = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(17)}}{2(1)} = \frac{8 \pm \sqrt{-4}}{2} = \frac{8 \pm 2i}{2} = 4 \pm i$$

For $\lambda = 4+i$, $(A - \lambda I)\vec{K} = \vec{0}$

$$\begin{bmatrix} 6-(4+i) & -1 \\ 5 & 2-(4+i) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\left[\begin{array}{cc|c} 2-i & -1 & 0 \\ \cancel{5} & \cancel{-(2+i)} & \cancel{0} \end{array}\right] \qquad R_2 = (2+i)R_1$$

$$(2-i)u_1 - u_2 = 0$$

$$u_2 = (2-i)\underbrace{u_1}_{FV=1}$$

$$\Rightarrow \vec{K} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2-i \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + i\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\vec{z} = \vec{K}e^{\lambda t} = \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + i\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) e^{(4+i)t} = \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + i\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) e^{4t}(\cos(t) + i\sin(t))$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{4t}\cos(t) + i\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{4t}\sin(t) + i\begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{4t}\cos(t) - \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{4t}\sin(t)$$

$$\vec{x}_1 = \operatorname{Re}\vec{z} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{4t}\cos(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{4t}\sin(t)$$

$$\vec{x}_2 = \operatorname{Im}\vec{z} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{4t}\sin(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{4t}\cos(t)$$

$$\Rightarrow \vec{x}(t) = c_1 \begin{bmatrix} \cos(t) \\ 2\cos(t) + \sin(t) \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} \sin(t) \\ 2\sin(t) - \cos(t) \end{bmatrix} e^{4t}$$

Bonus (10 points): Use variation of parameters to find a particular solution $\vec{x}_p$ for the nonhomogeneous system $\vec{x}' = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}\vec{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}e^t$, where $\Phi(t) = \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix}$ is a fundamental matrix of the associated homogeneous system.

$$\det \Phi(t) = 2e^{3t} - e^{3t} = e^{3t}$$

$$\Phi^{-1}(t) = \frac{1}{e^{3t}}\begin{bmatrix} e^{2t} & -e^{2t} \\ -e^t & 2e^t \end{bmatrix} = \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{bmatrix}$$

$$\vec{x}_p = \Phi(t)\int \Phi^{-1}(t)\vec{F}\,dt$$

$$= \Phi(t)\int \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{bmatrix}\begin{bmatrix} e^t \\ -e^t \end{bmatrix} dt$$

$$= \Phi(t)\int \begin{bmatrix} 1-(-1) \\ -e^{-t} - 2e^{-t} \end{bmatrix} dt$$

$$= \Phi(t)\int \begin{bmatrix} 2 \\ -3e^{-t} \end{bmatrix} dt$$

$$= \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix}\begin{bmatrix} 2t \\ 3e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} 4te^t + 3e^t \\ 2te^t + 3e^t \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 2 \end{bmatrix}te^t + \begin{bmatrix} 3 \\ 3 \end{bmatrix}e^t$$