MTH 204
Spring 2006
Exam 4

Name: Key

Section: D/E.

Read the directions carefully.
Write neatly in pencil and show all your work
(you will only get credit for what you put on paper).
Please do not share calculators during the test.
Each question is worth 20 points
DO NOT USE Decimals in any intermediate step.
You must calculate all eigenvalues and eigenvectors
BY HAND
The last page contains your Laplace tables.
If you have trouble during the test, feel free to ask me for help.

Score:________

1. Find the Laplace transform of the following functions.

A. $f(t) = te^{-3t}\cos(3t)$

$$\mathcal{L}\{te^{-3t}\cos(3t)\} = -\frac{d}{ds}\mathcal{L}\{e^{-3t}\cos(3t)\}$$

$$= -\frac{d}{ds}\mathcal{L}\{\cos(3t)\}_{s \to s-(-3)}$$

$$= -\frac{d}{ds}\left[\frac{s}{s^2+9}\bigg|_{s \to s+3}\right]$$

$$= -\frac{d}{ds}\left[\frac{s+3}{(s+3)^2+9}\right]$$

$$= -\frac{\left(((s+3)^2+9)(1) - 2(s+3)^2\right)}{((s+3)^2+9)^2}$$

$$= \frac{(s+3)^2 - 9}{((s+3)^2+9)^2}$$

$$= \frac{s^2+6s}{((s+3)^2+9)^2}$$

B. $g(t) = t^2 * te^t$

$$\mathcal{L}\{t^2 * te^t\} = \mathcal{L}\{t^2\}\mathcal{L}\{te^t\}$$

$$= \mathcal{L}\{t^2\}\mathcal{L}\{t\}|_{s \to s-1}$$

$$= \frac{2}{s^3}\left[\frac{1}{s^2}\bigg|_{s \to s-1}\right]$$

$$= \frac{2}{s^3(s-1)^2}$$

2. Solve the initial value problem $y''+y=\delta(t-\frac{3\pi}{2})$, $\begin{cases} y(0)=0 \\ y'(0)=1 \end{cases}$.

$$\mathcal{L}\{y''+y\}=\mathcal{L}\{\delta(t-\tfrac{3\pi}{2})\}$$

$$\mathcal{L}\{y''\}+\mathcal{L}\{y\}=\mathcal{L}\{\delta(t-\tfrac{3\pi}{2})\}$$

$$s^2Y(s)-sy(0)-y'(0)+Y(s)=e^{-\frac{3\pi}{2}s}$$

$$(s^2+1)Y(s)-0-1=e^{-\frac{3\pi}{2}s}$$

$$(s^2+1)Y(s)=e^{-\frac{3\pi}{2}s}+1$$

$$Y(s)=\frac{e^{-\frac{3\pi}{2}s}}{s^2+1}+\frac{1}{s^2+1}$$

$$F(s)=\frac{1}{s^2+1} \Rightarrow f(t)=\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$
$$=\sin(t)$$

Then $y(t)=\mathcal{L}^{-1}\{Y(s)\}$

$$=\mathcal{L}^{-1}\{e^{-\frac{3\pi}{2}s}F(s)+F(s)\}$$

$$=\mathcal{L}^{-1}\{e^{-\frac{3\pi s}{2}}F(s)\}+\mathcal{L}^{-1}\{F(s)\}$$

$$=\sin(t-\tfrac{3\pi}{2})\mathcal{U}(t-\tfrac{3\pi}{2})+\sin(t)$$

$$=\sin(t)+\cos(t)\,\mathcal{U}(t-\tfrac{3\pi}{2})$$

3. Use the Laplace transform to solve the integral equation $f(t) = t - \int_0^t (t-\tau)f(\tau)d\tau$.

$$f(t) = t - \int_0^t (t-\tau) f(\tau) d\tau$$

$$g(t-\tau) = t-\tau \Rightarrow g(t) = t$$

$$= t - (f(t) * t).$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t - (f(t)*t)\}$$

$$= \mathcal{L}\{t\} - \mathcal{L}\{f(t)*t\}$$

$$= \mathcal{L}\{t\} - \mathcal{L}\{f(t)\}\mathcal{L}\{t\}$$

$$F(s) = \frac{1}{s^2} - \frac{F(s)}{s^2}$$

$$F(s) + \frac{1}{s^2} F(s) = \frac{1}{s^2}$$

$$\left(\frac{s^2+1}{s^2}\right) F(s) = \frac{1}{s^2}$$

$$F(s) = \frac{1}{s^2+1}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \sin(t)$$

4. Find the general solution to the following homogeneous linear system $\vec{x}' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \vec{x}$.

$$0 = \det(A - \lambda I)$$
$$= \begin{vmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix}$$
$$= (3-\lambda)(-1-\lambda) + 4$$
$$= \lambda^2 - 2\lambda + 1$$
$$= (\lambda - 1)^2$$
$$\Rightarrow \lambda = 1, 1.$$

$$(A - \lambda I)\vec{K} = \vec{0}$$

$$\left[\begin{array}{cc|c} 3-1 & -4 & 0 \\ 1 & -1-1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} \cancel{2} & \cancel{-4} & \cancel{0} \\ 1 & -2 & 0 \end{array}\right] \Rightarrow k_1 - 2k_2 = 0$$

$$k_1 = 2\underbrace{k_2}_{FV = 1}$$

$$\Rightarrow \vec{K} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \vec{X}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t$$

$$(A - \lambda I)\vec{P} = \vec{K}$$

$$\left[\begin{array}{cc|c} 3-1 & -4 & 2 \\ 1 & -1-1 & 1 \end{array}\right] = \left[\begin{array}{cc|c} \cancel{2} & \cancel{-4} & \cancel{2} \\ 1 & -2 & 1 \end{array}\right] \Rightarrow p_1 - 2p_2 = 1$$

$$p_1 = 1 + 2\underbrace{p_2}_{FV = 0.}$$

$$\Rightarrow \vec{P} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \vec{X}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t$$

So our general solution is

$$\vec{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^t$$

5. Solve the following initial value problem $\vec{x}' = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \vec{x}$, $\vec{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$$0 = \det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 9 = \lambda^2 - 10\lambda + 16 = (\lambda-2)(\lambda-8) \Rightarrow \begin{cases} \lambda_1 = 2 \\ \lambda_2 = 8 \end{cases}$$

For $\lambda_1 = 2$, $(A - 2I)\vec{K}_1 = \vec{0}$

$$\left[\begin{array}{cc|c} 5-2 & 3 & 0 \\ 3 & 5-2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 3 & 3 & 0 \\ \cancel{3} & \cancel{3} & \cancel{0} \end{array}\right] \Rightarrow k_1 + k_2 = 0$$

$$-k_1 = k_2 \quad (k_1: \text{FV} = 1)$$

$$\Rightarrow \vec{K}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$$

For $\lambda_2 = 8$, $(A - 8I)\vec{K}_2 = \vec{0}$

$$\left[\begin{array}{cc|c} 5-8 & 3 & 0 \\ 3 & 5-8 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -3 & 3 & 0 \\ \cancel{3} & \cancel{-3} & \cancel{0} \end{array}\right] \Rightarrow -k_1 + k_2 = 0$$

$$k_1 = k_2 \quad (k_2: \text{FV} = 1)$$

$$\Rightarrow \vec{K}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{8t}$$

general solution $\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{8t}$

$$\vec{x}(0) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \begin{aligned} c_1 + c_2 &= 1 \\ -c_1 + c_2 &= -1 \end{aligned}$$

$$\Rightarrow c_2 = 0, \ c_1 = 1$$

So $\vec{x}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$

Bonus (10 points): Use the Laplace transform to solve the following system of IVPs

$$\frac{d^2x}{dt^2}+\frac{d^2y}{dt^2}=t^2 \qquad \frac{d^2x}{dt^2}-\frac{d^2y}{dt^2}=4t \qquad \text{subject to } \begin{cases} x(0)=8 & y(0)=0 \\ x'(0)=0 & y'(0)=0 \end{cases}$$

$$\mathcal{L}\{x''+y''\}=\mathcal{L}\{t^2\}$$
$$\mathcal{L}\{x''-y''\}=4\mathcal{L}\{t\}$$

$$s^2X(s)-s\overset{=8}{x(0)}-\overset{=0}{x'(0)}+s^2Y(s)-s\overset{=0}{y(0)}-\overset{=0}{y'(0)}=\frac{2}{s^3}$$
$$s^2X(s)-sx(0)-x'(0)-(s^2Y(s)-sy(0)-y'(0))=\frac{4}{s^2}$$

$$\Rightarrow s^2X(s)+s^2Y(s)=\frac{2}{s^3}+8s$$
$$+\quad s^2X(s)-s^2Y(s)=\frac{4}{s^2}+8s$$
$$2s^2X(s)=\frac{2}{s^3}+\frac{4}{s^2}+16$$
$$X(s)=\frac{1}{s^5}+\frac{2}{s^4}+\frac{8}{s}$$
$$\Rightarrow x(t)=\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\}+2\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}+8\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$
$$=\mathcal{L}^{-1}\left\{\frac{4!}{4!}\frac{1}{s^{4+1}}\right\}+2\mathcal{L}^{-1}\left\{\frac{3!}{3!}\frac{1}{s^{3+1}}\right\}+8\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$
$$=\frac{1}{4!}t^4+\frac{2}{3!}t^3+8$$
$$=\frac{1}{24}t^4+\frac{1}{3}t^3+8$$

$$s^2X(s)+s^2Y(s)=\frac{2}{s^3}+8s$$
$$-(s^2X(s)-s^2Y(s)=\frac{4}{s^2}+8s)$$
$$2s^2Y(s)=\frac{2}{s^3}-\frac{4}{s^2}$$
$$Y(s)=\frac{1}{s^5}-\frac{2}{s^4}$$
$$\Rightarrow y(t)=\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\}-2\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}$$
$$=\frac{1}{4!}\mathcal{L}^{-1}\left\{\frac{4!}{s^{4+1}}\right\}-\frac{2}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{s^{3+1}}\right\}$$
$$=\frac{1}{24}t^4-\frac{1}{3}t^3$$

So
$$\begin{cases} x(t)=\frac{1}{24}t^4+\frac{1}{3}t^3+8 \\ y(t)=\frac{1}{24}t^4-\frac{1}{3}t^3 \end{cases}$$

TABLE OF LAPLACE TRANSFORMS

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
1. 1	$\dfrac{1}{s}$
2. t	$\dfrac{1}{s^2}$
3. t^n	$\dfrac{n!}{s^{n+1}}$, n a positive integer
4. $t^{-1/2}$	$\sqrt{\dfrac{\pi}{s}}$
5. $t^{1/2}$	$\dfrac{\sqrt{\pi}}{2s^{3/2}}$
6. t^{α}	$\dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$, $\alpha > -1$
7. $\sin kt$	$\dfrac{k}{s^2+k^2}$
8. $\cos kt$	$\dfrac{s}{s^2+k^2}$
9. $\sin^2 kt$	$\dfrac{2k^2}{s(s^2+4k^2)}$
10. $\cos^2 kt$	$\dfrac{s^2+2k^2}{s(s^2+4k^2)}$
11. e^{at}	$\dfrac{1}{s-a}$
12. $\sinh kt$	$\dfrac{k}{s^2-k^2}$
13. $\cosh kt$	$\dfrac{s}{s^2-k^2}$
14. $\sinh^2 kt$	$\dfrac{2k^2}{s(s^2-4k^2)}$
15. $\cosh^2 kt$	$\dfrac{s^2-2k^2}{s(s^2-4k^2)}$
16. te^{at}	$\dfrac{1}{(s-a)^2}$
17. $t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$, n a positive integer
18. $e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$
19. $e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
20. $e^{at}\sinh kt$	$\dfrac{k}{(s-a)^2-k^2}$
21. $e^{at}\cosh kt$	$\dfrac{s-a}{(s-a)^2-k^2}$
22. $t\sin kt$	$\dfrac{2ks}{(s^2+k^2)^2}$
23. $t\cos kt$	$\dfrac{s^2-k^2}{(s^2+k^2)^2}$
24. $\sin kt + kt\cos kt$	$\dfrac{2ks^2}{(s^2+k^2)^2}$
25. $\sin kt - kt\cos kt$	$\dfrac{2k^3}{(s^2+k^2)^2}$
26. $t\sinh kt$	$\dfrac{2ks}{(s^2-k^2)^2}$
27. $t\cosh kt$	$\dfrac{s^2+k^2}{(s^2-k^2)^2}$
28. $\dfrac{e^{at}-e^{bt}}{a-b}$	$\dfrac{1}{(s-a)(s-b)}$
29. $\dfrac{ae^{at}-be^{bt}}{a-b}$	$\dfrac{s}{(s-a)(s-b)}$
30. $1-\cos kt$	$\dfrac{k^2}{s(s^2+k^2)}$
31. $kt-\sin kt$	$\dfrac{k^3}{s^2(s^2+k^2)}$
32. $\dfrac{a\sin bt - b\sin at}{ab(a^2-b^2)}$	$\dfrac{1}{(s^2+a^2)(s^2+b^2)}$
33. $\dfrac{\cos bt - \cos at}{a^2-b^2}$	$\dfrac{s}{(s^2+a^2)(s^2+b^2)}$
34. $\sin kt\sinh kt$	$\dfrac{2k^2 s}{s^4+4k^4}$
35. $\sin kt\cosh kt$	$\dfrac{k(s^2+2k^2)}{s^4+4k^4}$
36. $\cos kt\sinh kt$	$\dfrac{k(s^2-2k^2)}{s^4+4k^4}$
37. $\cos kt\cosh kt$	$\dfrac{s^3}{s^4+4k^4}$
38. $J_0(kt)$	$\dfrac{1}{\sqrt{s^2+k^2}}$

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
39. $\dfrac{e^{bt} - e^{at}}{t}$	$\ln \dfrac{s-a}{s-b}$
40. $\dfrac{2(1 - \cos kt)}{t}$	$\ln \dfrac{s^2 + k^2}{s^2}$
41. $\dfrac{2(1 - \cosh kt)}{t}$	$\ln \dfrac{s^2 - k^2}{s^2}$
42. $\dfrac{\sin at}{t}$	$\arctan\left(\dfrac{a}{s}\right)$
43. $\dfrac{\sin at \cos bt}{t}$	$\dfrac{1}{2}\arctan\dfrac{a+b}{s} + \dfrac{1}{2}\arctan\dfrac{a-b}{s}$
44. $\dfrac{1}{\sqrt{\pi t}}e^{-a^2/4t}$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}}$
45. $\dfrac{a}{2\sqrt{\pi t^3}}e^{-a^2/4t}$	$e^{-a\sqrt{s}}$
46. $\operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{s}$
47. $2\sqrt{\dfrac{t}{\pi}}e^{-a^2/4t} - a\operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{s\sqrt{s}}$
48. $e^{ab}e^{b^2t}\operatorname{erfc}\left(b\sqrt{t} + \dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}(\sqrt{s} + b)}$
49. $-e^{ab}e^{b^2t}\operatorname{erfc}\left(b\sqrt{t} + \dfrac{a}{2\sqrt{t}}\right) + \operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{be^{-a\sqrt{s}}}{s(\sqrt{s} + b)}$
50. $\delta(t)$	1
51. $\delta(t - t_0)$	e^{-st_0}
52. $e^{at}f(t)$	$F(s - a)$
53. $f(t - a)\mathscr{U}(t - a)$	$e^{-as}F(s)$
54. $\mathscr{U}(t - a)$	$\dfrac{e^{-as}}{s}$
55. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
56. $t^nf(t)$	$(-1)^n\dfrac{d^n}{ds^n}F(s)$
57. $\int_0^t f(\tau)g(t - \tau)\,d\tau$	$F(s)G(s)$