My general research area is the study of differential and difference equations. Currently I am working in an emerging field in dynamical systems. I would describe my work as a cross between the theoretical and applied. The goal of my research is to unify and extend continuous and discrete analysis into a more general theory. Specifically, my work has great potential for the study of hybrid systems sampled with continuous, discrete, or irregular measurements. Computationally, such hybrid systems have their advantages in studying a number of real world applications. In particular, I am interested in applications in control engineering, game theory, forecasting, and mathematical finance.

1. Overview

My main research focus is the study of optimal control and estimation in continuous and discrete time. More specifically, I study such processes on dynamic equations on time scales. A time scale, denoted by $\mathbb{T}$, is a nonempty closed subset of the reals. In order to understand the structure of a time scale, we define the following operators on $\mathbb{T}$. We define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\}$ and the graininess function $\mu : \mathbb{T} \to [0, \infty)$ by $\mu(t) = \sigma(t) - t$. Here σ represents the shift to the next available point in $\mathbb{T}$ while μ represents the distance between two consecutive points in $\mathbb{T}$. For any function $f : \mathbb{T} \to \mathbb{R}$, we often define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by $f^\sigma = f \circ \sigma$. Standard calculus operations can be defined on a time scale, namely Δ-differentiation and Δ-integration. The table below offers some examples of time scales and standard operations on them. A more detailed introduction can be found in [2, 3].

$\mathbb{T}$	$\mu(t)$	$\sigma(t)$	$f^\Delta(t)$	$\int_a^b f(\tau)\Delta\tau$
$\mathbb{R}$	0	t	$f'(t)$	$\int_a^b f(\tau)d\tau$
$\mathbb{Z}$	1	$t+1$	$\Delta f(t) = f(t+1) - f(t)$	$\int_a^b f(t)\,\Delta t = \sum_{\tau=a}^{b-1} f(\tau)$
$h\mathbb{Z}$	h	$t+h$	$f^\Delta(t) = \frac{f(t+h)-f(t)}{h}$	$\int_a^b f(t)\,\Delta t = \sum_{k=a/h}^{b/h-1} hf(kh)$
$\overline{q^\mathbb{Z}}$	$(q-1)t$	qt	$D_q f(t) := \frac{f(qt)-f(t)}{(q-1)t}$	$\int_a^b f(t)\,\Delta t = \sum_{\tau=a}^{b}(q-1)\tau f(\tau)$

Control theory lends itself well such unification, as the structure and behavior of discrete control processes closely mirror their continuous counterparts. Consequently, the study of control systems on time scales can be thought of as a generalized sampling technique that allows us to evaluate processes with continuous, discrete, or possibility uneven measurements. Due to the infancy of this field, there are still a number of open problems. I will describe my contributions to this field as well as future plans in detail below.

2. Past Research

2.1. **Controllability and Observability.** Working under Dr Martin Bohner, we first extended well known results in controllability and observability for linear time invariant systems on time scales in [6]. R.E. Kalman introduced these concepts in the early 1960s for the continuous and discrete time cases (see [13,16]). Since then, they have become the backbone of modern control theory with many applications in engineering and mathematics. Here we studied the dynamic system

$$\begin{aligned} x^{\Delta} &= -Ax^{\sigma} + Bu \\ y &= Cx, \end{aligned}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input (control), and $y \in \mathbb{R}^r$ is the output. Then given dynamic system is said to be controllable if and only if the matrix

$$\Gamma_C[A, B] := \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

has full rank n while it is said to be observable if and only if

$$\Gamma_O[A, C] := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has rank n. These are the same controllability and observability matrices in the discrete and continuous cases. Also, these concepts are mathematical duals of each other in the discrete and continuous cases. This relationship is preserved in their unification on time scales. There have been a number of other attempts to examine these concepts for a similar system (see [1,8,9]). As it turns out, there seems to be a trade off between these two systems, which means it may be more beneficial to use one system over the other depending on whether we are using the controllability or observability property. This is due to the way the matrix exponential is defined on time scales.

2.2. **The Linear Quadratic Regulator (LQR) and Tracker (LQT).** Much of my work has centered around the formulation and solution of a new kind of optimal control problem. The goal of such problems to find an optimal control that minimizes a quadratic cost functional associated with some dynamic system. Kalman and Koepcke [17] first used this method in the discrete case. Shortly thereafter, Kalman extended these results to continuous time (see [11,14]). Since then, what is now called the linear quadratic regulator (LQR) plays a central role in control engineering. We have generalized this method (see [7]) to consider

$$x^{\Delta}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0$$

associated with the quadratic cost functional

$$J = \frac{1}{2}x^T(t_{\mathrm{f}})S(t_{\mathrm{f}})x(t_{\mathrm{f}}) + \frac{1}{2}\int_{t_0}^{t_{\mathrm{f}}}(x^TQx + u^TRu)(\tau)\Delta\tau,$$

where $S(t_{\mathrm{f}})$, $Q \geq 0$ and $R > 0$. Depending on the final state the optimal control can take two different forms. If the final state is fixed, the (open-loop) optimal control mirrors our controllability criterion results we obtained in [6]. On the other hand if the final state is free, then the (closed-loop) optimal control can be written as $u^*(t) = -K(t)x(t)$, where K represents a generalized state feedback gain.

Next we extended our results on the LQR (see [5]) to linear quadratic tracking on time scales. Here we determined an affine optimal control that forces our our dynamic system to track a desired reference trajectory over a fixed time. In this setting, we found an optimal control that contains a feedback term as well as a feedforward term that anticipates this desired final trajectory. In this paper we also examined the numerical advantages time scales offers to control theory. When the dynamics are stationary and the isolated time scale is known in advance, our tracking algorithms incorporate the gaps between sampling times within the equations of our states, inputs, and gains. This also offers an advantage when we need schedule our gains, a useful tool in radar analysis. Other potential applications of the LQT on time scales include disturbance-rejection, heat dynamics, game theory, and economics.

2.3. **The Kalman Filter.** We have also generalized the Kalman filter to dynamic equations on time scales. This optimal estimation method was first tackled by Kalman [12] in the discrete case and later with Bucy [15] in the continuous case in the early 1960s. The effect of this filter cannot be understated. Following Kalman's visit to the NASA Ames Research Center, Kalman's filter was applied to the problem of navigating to the moon with the Apollo program. In [4], we considered the linear stochastic dynamic system

$$\begin{aligned} x^{\Delta}(t) &= Ax(t) + Bu(t) + Gw(t), \quad x(t_0) = x_0, \\ y(t) &= Cx(t) + v(t), \end{aligned}$$

where state and output are influenced by unknown disturbances w and v. In this setting, we want an unbiased estimate, $\hat{x}$, that ensures the smallest error covariance possible. We made the assumptions that our measurement are in "real time" and that the intial estimate is equal to the initial state mean (i.e., $\hat{x}(t_0) = \overline{x}(t_0)$) so that the observer is unbiased and mirrors the results for the Kalman-Bucy filter. We proved that this observer is given by

$$\hat{x}^{\Delta}(t) = A\hat{x}(t) + Bu(t) + L(t)[y(t) - C\hat{x}(t)], \quad \hat{x}(t_0) = \overline{x}(t_0),$$

where L is a Kalman gain. Finally, we have made an argument that LQR and the Kalman are mathematical duals to each other, as the Riccati equations and gains that describe each optimal problem look very similar to each other. This relationship exists in both the discrete and continuous cases, and is preserved in their unification on time scales.

3. Current Research and Future Plans

Looking forward, there are a number of open problems based on my work thus far. Below I list some of my current research projects and future plans.

3.1. **Extensions to the LQR.** Currently, I am working on extending my results of the LQR to other applied problems. One such extension is optimal pursuit-evasion games. These is a two player game first considered by Ho, Bryson, and Baron in the continuous case (see [10]). The pursuing state seeks to intercept the evading state at time t_{f} while the latter state seeks to do the opposite. This is essentially a minimax problem. I am also working on extending these results to solve minimum time problems, regulators with cross coupling terms and a prescribed degree of stability, and bang bang control.

3.2. **Bilinear Systems.** To best of my knowledge, control theory for nonlinear dynamic systems on time scales has not been investigated in great detail. Currently I am studying a special case of nonlinear dynamic systems given by

$$\begin{aligned} x^{\Delta} &= Ax + Dxu + bu \\ y &= cx, \end{aligned}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$. This system has been well studied in the discrete and continuous cases. Some potential applications for these systems on time scales include nuclear reactor systems, suspension systems, furmentation processes, and heat exchange systems. I am also interested in extending my results on the LQR and Kalman filter for bilinear systems on time scale as these systems should give us insight to solve more general nonlinear dynamic systems.

3.3. **Other Interests.** While my work thus far has solely been in the time domain, I am also interested in studying similar problems in the frequency domain. There are many engineering problems that involve both discrete and continuous signals, or signals that are sampled at irregular points in time. Therefore, signal processing on time scales has potential for many applications, such as radar systems and electronic warfare. Thus far there has been little, if any, study of signal processing on time scales.

My research interests are compatible with those who work in dynamical systems, time series, dynamic programming, and adaptive control among others. Also from a pedological viewpoint, the theory of time scales has a lot to offer. In unifying concepts from the discrete and continuous cases, students will gain a deeper understanding and appreciation of both. With any research I work on, I would like to collaborate with researchers in other departments on campus.

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