MTH 204
Quiz 5
21 Oct 2005

Name: Key
Section: A/C

Follow the directions carefully. You must show all your work in order to get full credit. Do not share calculators. Please write in pencil.

LEAD Session - Mondays
6PM, Rolla G4.

1. Consider the DE $x^2y'' - xy' + y = 2x$

a. Classify the DE by order, linearity, coefficients, and state whether or not the DE is homogeneous.

b. What method(s) do you use to solve the DE?

c. Solve the DE.

a. 2nd order, linear, variable coefficients, and nonhomogeneous.

b. Variation of Parameters

c. Solve for $y_h(x)$ first

$x^2y'' - xy' + y = 0 \quad , x>0$

Assume $y(x) = x^r$

$y'(x) = rx^{r-1}$

$y''(x) = r(r-1)x^{r-2}$

$x^2r(r-1)x^{r-2} - x(rx^{r-1}) + x^r = 0$

$x^r[r(r-1) - r + 1] = 0$

$x^r[r^2 - 2r + 1] = 0$

$(r-1)^2 = 0 \quad$ since $x \neq 0$

$r = 1, 1$

$y_1(x) = x$

$y_2(x) = x\ln x$

$$W(y_1, y_2) = \begin{vmatrix} x & x\ln x \\ 1 & \ln x + 1 \end{vmatrix} = x\ln x + x - x\ln x = x$$

Put DE in standard form: $y'' - \frac{1}{x}y' + \frac{1}{x^2}y = \frac{2}{x}$

Assume $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$

$$W_1 = \begin{vmatrix} 0 & x\ln x \\ 2x^{-1} & \ln x + 1 \end{vmatrix} = -2\ln x \qquad W_2 = \begin{vmatrix} x & 0 \\ 1 & 2x^{-1} \end{vmatrix} = 2$$

$$u_1'(x) = \frac{W_1}{W(y_1, y_2)} = \frac{-2\ln x}{x} \qquad u_2'(x) = \frac{W_2}{W(y_1, y_2)} = \frac{2}{x}$$

$$u_1(x) = -2\int \frac{\ln x}{x}\,dx = -v^2 = -(\ln x)^2 \qquad u_2(x) = 2\int \frac{dx}{x} = 2\ln x$$

$v = \ln x$

$dv = \frac{1}{x}dx$

$$y_p = u_1y_1 + u_2y_2 = -x(\ln x)^2 + (2\ln x)(x\ln x) = -x(\ln x)^2 + 2x(\ln x)^2 = x(\ln x)^2$$

So $y(x) = y_h + y_p = c_1x + c_2x\ln x + x(\ln x)^2$

2. Solve the following the system of DEs.

$$\frac{dx}{dt} = 4x + 7y \quad \Rightarrow \quad \frac{dx}{dt} - 4x - 7y = 0$$

$$\frac{dy}{dt} = x - 2y \qquad\qquad -x + \frac{dy}{dt} + 2y = 0$$

$$(D-4)x - 7y = 0$$

$$-x + (D+2)y = 0 \quad \Rightarrow \quad x(t) = (D+2)y$$

So $(D-4)(D+2)y - 7y = 0$

$((D-4)(D+2) - 7)y = 0$

$(D^2 + 2D - 4D - 8 - 7)y = 0$

$(D^2 - 2D - 15)y = 0$

$(D-5)(D+3)y = 0$ Assume $y(t) = e^{rt}$

$\Rightarrow (r+3)(r-5) = 0$

$r = 5, -3$

$y(t) = c_1 e^{5t} + c_2 e^{-3t}$

$x(t) = (D+2)y = y' + 2y$

$y'(t) = 5c_1 e^{5t} - 3c_2 e^{-3t}$

$x(t) = 5c_1 e^{5t} - 3c_2 e^{-3t} + 2(c_1 e^{5t} + c_2 e^{-3t})$

$= 7c_1 e^{5t} - c_2 e^{-3t}$

So the solution to the system is

$$\begin{cases} x(t) = 7c_1 e^{5t} - c_2 e^{-3t} \\ y(t) = c_1 e^{5t} + c_2 e^{-3t} \end{cases}$$