MTH 204
Quiz 6
10 March 2006

Name: Key
Section: D\E

Follow the directions carefully. Remember to show all your work in order to get full credit. Please write neatly in pencil. If you have any trouble, feel free to ask me for help.

LEAD Session - Wednesdays
6:00-7:30 PM
CSF G5D

Test 3: 24 March
4.5-

1 Consider the DE $x^2y'' - 4xy' = x^5$; $x>0$

a. Classify the DE.

2nd order, linear, variable coefficients, nonhomogeneous (Cauchy-Euler)

b. What method(s) do you have to solve the DE?

VOP, $x=e^t \Rightarrow$ MUC

c. Solve the DE.

$x^2y'' - 4xy' = 0$ Assume $y(x) = x^m$

$x^2[m(m-1)x^{m-2}] - 4x[mx^{m-1}] = 0$

$x^m[m(m-1) - 4m] = 0$

$\Rightarrow m^2 - 5m = 0$

$m_1 = 0, m_2 = 5$

$y_h = c_1 + c_2x^5$

Method 1: VOP

$y'' - \dfrac{4}{x}y' = x^3$

$$W(y_1,y_2) = \begin{vmatrix} 1 & x^5 \\ 0 & 5x^4 \end{vmatrix} = 5x^4$$

Assume $y_p = u_1y_1 + u_2y_2$

$$W_1 = \begin{vmatrix} 0 & x^5 \\ x^3 & 5x^4 \end{vmatrix} = -x^8 \qquad W_2 = \begin{vmatrix} 1 & 0 \\ 0 & x^3 \end{vmatrix} = x^3$$

$$u_1' = \frac{W_1}{W(y_1,y_2)} = \frac{-x^8}{5x^4} = \frac{-x^4}{5} \qquad u_2' = \frac{W_2}{W(y_1,y_2)} = \frac{x^3}{5x^4} = \frac{1}{5x}$$

$$u_1(x) = \frac{-1}{5}\int x^4\,dx = \frac{-x^5}{25} \qquad u_2(x) = \frac{1}{5}\int \frac{dx}{x} = \frac{1}{5}\ln(x)$$

$$\Rightarrow y_p = \frac{-x^5}{25} + \frac{1}{5}x^5\ln(x)$$

$\frac{-x^5}{25} \hookrightarrow y_h$

$$= \frac{1}{5}x^5\ln(x)$$

So $y(x) = c_1 + c_2x^5 + \frac{1}{5}x^5\ln(x)$

<u>Method 2</u>: MUC

Assume $x = e^t$

$$\Rightarrow \left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) - 4\frac{dy}{dt} = (e^t)^5 = e^{5t}$$

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} = e^{5t}$$

Solve $\frac{d^2y}{dt^2} - 5\frac{dy}{dt} = 0$ Assume $y(t) = e^{rt}$

$\Rightarrow r^2 - 5r = 0$

$r_1 = 0 \quad r_2 = 5$

$y_1 = 1$

$y_2 = e^{5t}$

$y_h = c_1 + c_2e^{5t}$

$g(t) = e^{5t} \qquad r = 5 \Rightarrow r - 5 = 0$

$\Rightarrow (D-5)g(t) = 0$

$(D^2-5D)y = e^{5t}$

$\Rightarrow (D-5)D(D-5)y = (D-5)[e^{5t}] = 0$

Assume $y(t) = e^{rt}$

$\Rightarrow (r-5)r(r-5) = 0$

$r = 5, 0, 5$

$$y(t) = \underbrace{c_1 e^{5t} + c_2}_{y_h} + \underbrace{c_3 t e^{5t}}_{y_p}$$

$y_p = Ate^{5t}$

$y_p' = Ae^{5t} + 5Ate^{5t}$

$y_p'' = 5Ae^{5t} + 5Ae^{5t} + 25Ate^{5t}$

$= 10Ae^{5t} + 25Ate^{5t}$

$y_p'' - 5y_p' = (25A - 5(5A))te^{5t} + (10A - 5A)e^{5t}$

$= 5Ae^{5t}$

$= e^{5t}$

$\Rightarrow A = 1/5$

$y_p = \frac{1}{5}te^{5t}$

Then $y(t) = c_1 + c_2 e^{5t} + \frac{1}{5}te^{5t}$

$t = \ln(x) \Rightarrow y(x) = c_1 + c_2 e^{5\ln x} + \frac{1}{5}(\ln x)(e^{5\ln x})$

$= c_1 + c_2 x^5 + \frac{1}{5}x^5 \ln(x)$

2. A 4 ft spring measures 8 ft long after an object weighing 8 lbs is attached to it. The medium through which the object moves offers a damping force numerically equal to the instantaneous velocity. Find the equation of motion if the object is released from equilibrium position with a downward velocity of 5 ft/s.

$m = \frac{W}{g} = \frac{8}{32} = \frac{1}{4}$

$Ks = mg \Rightarrow 4K = 8$

$K = 2$

$\beta = 1$

$\Rightarrow \frac{1}{4}y'' + y' + 2y = 0 \qquad \begin{cases} y(0) = 0 \\ y'(0) = 5 \end{cases}$

Assume $y(t) = e^{rt}$

$\Rightarrow \frac{1}{4}r^2 + r + 2 = 0$

$r^2 + 4r + 8 = 0$

$r = \frac{-4 \pm \sqrt{16 - 4(8)}}{2(1)} = \frac{-4 \pm 4i}{2} = -2 \pm 2i$

$y(t) = e^{-2t}[c_1 \cos(2t) + c_2 \sin(2t)]$

$y(0) = 1[c_1 \cos(0) + c_2 \sin(0)] = c_1 = 0$

$\Rightarrow y(t) = c_2 e^{-2t} \sin(2t)$

$y'(t) = -2c_2 e^{-2t} \sin(2t) + 2c_2 e^{-2t} \cos(2t)$

$y'(0) = 0 + 2c_2 = 5 \quad \Rightarrow c_2 = 5/2$

$\Rightarrow y(t) = \frac{5}{2} e^{-2t} \sin(2t)$