Review of Integration Techniques

A. Integration by Parts: If u and v are functions of x and have continuous derivatives, then

$\int u\,dv = uv - \int v\,du.$

Guidelines for picking u

Inverse trigonometric functions ($\arctan(x)$, $\operatorname{arcsec}(x)$, etc)
Logarithms ($\ln(x)$)
Algebraic functions (polynomials: $3x^2$, $x^{10}-x$, etc)
Trigonometric functions ($\sin(x)$, $\cos(3x)$, $\tan(2x)$, etc)
Exponential functions (e^x, 2^x, etc).

Ex $\int x\sin(x)dx = -x\cos(x) - \int(-\cos(x))dx$ $u = x$ $dv = \sin(x)$

$= -x\cos(x) + \int\cos(x)dx$ $du = 1$ $v = -\cos(x)$

$= -x\cos(x) + \sin(x) + C$

B. Partial Fractions

Decomposition of $\frac{N(x)}{D(x)}$ into Partial Fractions

1. Divide if improper: If degree of $N(x) \geq$ degree of $D(x)$, then divide to get

$\frac{N(x)}{D(x)} = (\text{a polynomial}) + \frac{N_1(x)}{D(x)}$ where degree of $N_1(x)$ < degree of $D(x)$

2. Factor $D(x)$: Factor the denominator into factors of the form

$(px+q)^m$ (linear)

$(ax^2+bx+c)^n$ (quadratic) where ax^2+bx+c is irreducible

3. Linear factors: For each factor of the form $(px+q)^m$, the partial fraction decomposition must include the following sum of m fractions,

$$\frac{A_1}{px+q} + \frac{A_2}{(px+q)^2} + \dots + \frac{A_m}{(px+q)^m}$$

4. Quadratic factors: For each factor of the form $(ax^2+bx+c)^n$, the partial fraction decomposition must include the following sum of n fractions

$$\frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \dots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

Ex $\displaystyle\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx$

$$\frac{5x^2+20x+6}{x^3+2x^2+x} = \frac{5x^2+20x+6}{x(x+1)^2}$$

$$= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

Now multiply both sides by the least common denominator to get

$$5x^2+20x+6 = A(x+1)^2 + Bx(x+1) + Cx$$

To solve for A, let $x=0$

$$6 = A(1) + 0 + 0 \implies A = 6$$

To solve for C, let $x=-1$

$$5-20+6 = -9 = 0+0-C \implies C=9$$

To solve for B, let $x=1$

$$5+20+6 = A(4)+B(2)+C$$

$$31 = 6(4)+2B+9$$

$$-2 = 2B \qquad \Longrightarrow B=-1$$

Now $\displaystyle\int \frac{5x^2+20x+6}{x(x+1)^2}\,dx = \int\left(\frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}\right)dx$

$$= 6\int\frac{dx}{x} - \int\frac{dx}{x+1} + 9\int\frac{dx}{(x+1)^2}$$

$$= 6\ln|x| - \ln|x+1| + \frac{9(x+1)^{-1}}{-1} + C$$

$$= \ln|x^6| - \ln|x+1| - \frac{9}{x+1} + C$$

$$= \ln\left|\frac{x^6}{x+1}\right| - \frac{9}{x+1} + C$$