This is a closed-book, closed-notes exam. The only items you are allowed to use are writing implements. Mark each sheet of paper you use with your name and clearly indicate the problem number.

The max number of points per question is indicated in square brackets after each question. The sum of the max points for all the questions is 55, but note that the max exam score will be capped at 50 (i.e., there are 5 bonus points, but you cant score more than 100%). Partial credit will be awarded, so show your work!

You have exactly 60 minutes to complete this exam. Keep your answers clear and concise while complete. Good luck!

1. Let $S_1 = \{1, 3, 5\}, S_2 = \{9, 7\}$, and $S_3 = \{2, 4, 8, 6, 8\}$.

   (a) (3 points) What is $\bigcup_{i=1}^{3} S_i$?

   Solution: $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

   (b) (3 points) What is $\bigcap_{i=1}^{3} S_i$?

   Solution: $\emptyset$

   (c) (2 points) Do S_1, S_2, and S_3 form a partition? Why or why not? If so, state the set of which they are a partition.

   Solution: Yes: they are pairwise disjoint ($S_1 \cap S_2 = \emptyset, S_1 \cap S_3 = \emptyset, S_2 \cap S_3 = \emptyset\}$). The set they partition is $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

2. (5 points) Let $A = \{a, b, c\}$ and let B be the set recursively defined by the following rules:

   1. For all $x \in A$, $\{x\} \in B$ (i.e., the set containing x is in B).
   2. If $s, t \in B$, then $s \cup t \in B$ and $s \cap t \in B$.
   3. No other elements are in B.

   Write the elements of B.

   Solution: This is the powerset of A: $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$.

3. Let $f : \mathbb{R} \mapsto \mathbb{Z}$ be the function defined by $f(x) = \lfloor x \rfloor$ — that is, f rounds x down to the nearest integer. For instance, $\lfloor 4.7 \rfloor = 4$ and $\lfloor 6 \rfloor = 6$.

   (a) (2 points) What is the domain and co-domain of f?

   Solution: Domain: $\mathbb{R}$; co-domain: $\mathbb{Z}$.

   (b) (4 points) Is f onto? Prove or give a counterexample.

   Solution: Yes. For any $n \in \mathbb{Z}$, $n = f(n)$.

   (c) (4 points) Is f one-to-one? Prove or give a counterexample.

> **Solution:** No! $f(3.5) = f(3)$, but $3.5 \neq 3$.

(d) (4 points) Is f a bijection? If so, find f^{-1}. If not, state why not.

> **Solution:** f is not one-to-one, so it is not a bijection.

4. Given the equivalence relation $A : \mathbb{Z} \mapsto \mathbb{Z}$ defined by $x \, A \, y$ if and only if $|x| = |y|$.

(a) (4 points) Write the elements of [5], [0], [−2], and [−5].

> **Solution:** $[5] = \{5, -5\}$, $[0] = \{0\}$, $[-2] = \{2, -2\}$, $[-5] = [5]$.

(b) (3 points) Describe the distinct equivalence classes of A.

> **Solution:** There is one for each non-negative integer; each equivalence class contains x and $-x$.

5. (8 points) Draw a Hasse diagram for the partial order

$$\begin{aligned} L = \{&(\epsilon,\epsilon),(\epsilon,a),(\epsilon,b),(\epsilon,aa),(\epsilon,ab),(\epsilon,ba),(\epsilon,bb),\\ &(a,a),(a,aa),(a,ab),\\ &(aa,aa),(aa,ab),\\ &(b,b),(b,ba),(b,bb),\\ &(ba,ba),(bb,bb)\} \end{aligned}$$

> **Solution:** Ack! This is not actually a partial order! Namely, it is not reflexive for ab. The pair (aa, ab) should be (ab, ab).
>
> If that change is made, the Hasse diagram for this order is an inverted binary tree with ϵ as its root, a and b as internal nodes, and aa, ab, ba, bb as leaves.

6. Consider the statement "$n^n > n!$ for all integers $n > n_0$".

(a) (3 points) What is the smallest integer n_0 this statement is true for?

> **Solution:** 2. $2^2 = 4 > 2! = 2$.

(b) (10 points) Prove this statement by induction.

> **Solution:** Base case: see previous part.
> Inductive Hypothesis: suppose $k^k > k!$ for arbitrary $k \in \mathbb{Z}$, $k \geq 2$.
> $(k+1)! = (k+1)k!$, and, by the inductive hypothesis, $(k+1)k! < (k+1)k^k$.
> $(k+1)k^k = kk^k + k^k = k^{k+1} + k^k$.
> $(k+1)^{k+1} = k^{k+1} + ck^k + \cdots + 1$ where $c \geq 1$ (in fact, $c = \binom{k}{1} = k$).
> (Alternatively: $(k+1)^{k+1} = (k+1)(k+1)^k > (k+1)k^k$.)
> Thus, $(k+1)! < (k+1)k^k < (k+1)^{k+1}$.
> Therefore, $n^n > n!$ for all integers $n \geq 2$.