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Oscillatory Behavior of Solutions of Third-Order Delay and Advanced Dynamic Equations

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Abstract

In this paper, we consider oscillation criteria for certain third-order delay and advanced dynamic equations on unbounded time scales. A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers. Examples will be given to illustrate some of the results.

Keywords: Oscillation; third-order; dynamic equations; time scales

Mathematics Subject Classification (2010): 34N05; 39A10; 39A21

1 Introduction

In this paper, we are concerned with oscillation criteria for solutions of the third-order delay and advanced dynamic equations

$$\left(\frac{1}{a(t)}\left(x^{\Delta}(t)\right)^{\alpha}\right)^{\Delta\Delta}+q(t)f(x[g(t)])=0 \tag{1.1}$$

and

$$\left(\frac{1}{a(t)}\left(x^{\Delta}(t)\right)^{\alpha}\right)^{\Delta\Delta}=q(t)f(x[g(t)])+p(t)h(x[k(t)]) \tag{1.2}$$

on $[t_0,\infty)_{\mathbb{T}}$ such that $t_0\in\mathbb{T}$ and $t_0\geq 0$, where α is the ratio of two positive odd integers, $a,p,q\in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},(0,\infty))$ with

$$\int^{\infty} a^{\frac{1}{\alpha}}(s)\Delta s=\infty, \tag{1.3}$$

and $g,k\in \mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{T})$ are nondecreasing functions such that $g(t)<t<k(t)$ and $\lim_{t\to\infty} g(t)=\infty$. We also assume that $f,h\in C(\mathbb{R},\mathbb{R})$ such that $xf(x)>0$, $xh(x)>0$, $f(x)$ and $h(x)$ are nondecreasing for $x\neq 0$ satisfying

$$-f(-xy)\geq f(xy)\geq f(x)f(y) \text{ if } xy>0 \tag{1.4}$$

and

$$-h(-xy)\geq h(xy)\geq h(x)h(y) \text{ if } xy>0. \tag{1.5}$$

By a solution of equation (1.1) (or (1.2)) we mean a function $x\in C^1_{rd}\left([T_x,\infty)_T,\mathbb{R}\right)$, $T_x\geq t_0$, which has the property that $(1/\alpha)\left(x^{\Delta}\right)^{\alpha}\in C^2_{rd}\left([T_x,\infty)_{\mathbb{T}},\mathbb{R}\right)$ and satisfies

(1.1) (or (1.2)) for all large $t \geq T_x$. A nontrivial solution is said to be nonoscillatory if it is eventually positive or eventually negative and it is oscillatory otherwise. A dynamic equation is said to be oscillatory if all its solutions are oscillatory.

Since we are interested in the oscillatory behavior of solutions of (1.1) and (1.2) near infinity, we assume throughout this paper that our time scale is unbounded above. An excellent introduction of time scales calculus can be found in the books by Bohner and Peterson [1] and [2].

The purpose of this paper is to extend the oscillation results given in [3] and [4]. Oscillation criteria for third order dynamic equations are recently studied in [5–8]. Other papers related with oscillation of higher order dynamic equations can be found in [9,10]. The well-known books concerning the oscillation theory are [11,12].

For simplification, we define the following operators

$$L_0x(t) = x(t), \quad L_1x(t) = \frac{1}{a(t)}\left(L_0^{\Delta}x(t)\right)^{\alpha}, \quad L_2x(t) = L_1^{\Delta}x(t), \quad L_3x(t) = L_2^{\Delta}x(t).$$

Thus (1.1) and (1.2) can be written as

$$L_3x(t) + q(t)f(x[g(t)]) = 0$$

and

$$L_3x(t) = q(t)f(x[g(t)]) + p(t)h(x[k(t)]),$$

respectively. In what follows we use the following notation. For $(t,s,T) \in [s,\infty)_{\mathbb{T}} \times [T,\infty)_{\mathbb{T}} \times [t_0,\infty)_{\mathbb{T}}$

$$A(t,s) = \int_s^t a^{\frac{1}{\alpha}}(u)(u-s)^{\frac{1}{\alpha}}\Delta u \tag{1.6}$$

and

$$B(t,s) = \int_s^t a^{\frac{1}{\alpha}}(u)(t-u)^{\frac{1}{\alpha}}\Delta u. \tag{1.7}$$

2 Oscillation Criteria for (1.1)

In this section, we investigate some oscillation criteria for solutions of the third-order delay equation (1.1).

Theorem 2.1 *Let* (1.3) *and* (1.4) *hold and assume that*

$$\frac{f(u^{\frac{1}{\alpha}})}{u} \geq c > 0 \tag{2.1}$$

for $u \neq 0$ *and a constant* c*. If for* $t \in [t_0,\infty)_{\mathbb{T}}$

$$\limsup_{t\to\infty} \int_{g(t)}^t q(s)f(A(g(s),t_0))\Delta s > \frac{1}{c} \tag{2.2}$$

and

$$\limsup_{t\to\infty} \int_{g(t)}^{t} q(s)f(B(g(t), g(s)))\Delta s > \frac{1}{c}, \tag{2.3}$$

where A and B are defined as in (1.6) *and* (1.7), *respectively, then* (1.1) *is oscillatory.*

Proof Let x be a nonoscillatory solution of (1.1) and assume that without loss of generality $x(t) > 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$ and so $L_3x(t) \le 0$ eventually for $t \in [t_0, \infty)_{\mathbb{T}}$. Therefore, there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $L_1x(t)$ and $L_2x(t)$ are of one sign for all $t \in [t_1, \infty)_{\mathbb{T}}$. We now distinguish the following two cases:

(I) $L_1x(t) > 0$ and $L_2x(t) > 0$ eventually;

(II) $L_1x(t) < 0$ and $L_2x(t) > 0$ eventually.

We now start with the first case.

(I) Assume that $L_1x(t) > 0$ and $L_2x(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Then we have

$$\int_{t_1}^{t} L_2x(s)\Delta s = L_1x(t) - L_1x(t_1) \le L_1x(t), \quad t \in [t_1, \infty)_{\mathbb{T}}.$$

Since L_2x is nonincreasing, we have

$$(t - t_1)L_2x(t) \le L_1x(t), \quad t \in [t_1, \infty)_{\mathbb{T}}.$$

This implies that

$$x^{\Delta}(t) \ge a^{\frac{1}{\alpha}}(t)(t - t_1)^{\frac{1}{\alpha}} L_2^{\frac{1}{\alpha}} x(t), \quad t \in [t_1, \infty)_{\mathbb{T}}.$$

Integrating the above inequality from t_1 to t, we have

$$x(t) \ge L_2^{\frac{1}{\alpha}} x(t) \int_{t_1}^{t} a^{\frac{1}{\alpha}}(s)(s - t_1)^{\frac{1}{\alpha}} \Delta s = A(t, t_1)L_2^{\frac{1}{\alpha}} x(t), \quad t \in [t_1, \infty)_{\mathbb{T}},$$

where A is defined as in (1.6). Hence there exists $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that

$$x[g(t)] \ge A(g(t), t_1)L_2^{\frac{1}{\alpha}} x[g(t)], \quad t \in [t_2, \infty)_{\mathbb{T}}. \tag{2.4}$$

From (2.4) and (1.4) in (1.1) we obtain that

$$-L_3x(t) \ge q(t)f(A(g(t), t_1))f(L_2^{\frac{1}{\alpha}} x[g(t)]), \quad t \in [t_2, \infty)_{\mathbb{T}}.$$

Integrating the above inequality from $g(t)$ to t, we obtain that

$$L_2x[g(t)] \ge \int_{g(t)}^{t} q(s)f(A(g(s), t_1))f(L_2^{\frac{1}{\alpha}} x[g(s)])\Delta s, \quad t \in [t_2, \infty)_{\mathbb{T}}$$

or

$$L_2x[g(t)] \ge f(L_2^{\frac{1}{\alpha}} x[g(t)]) \int_{g(t)}^{t} q(s)f(A(g(s), t_1))\Delta s, \quad t \in [t_2, \infty)_{\mathbb{T}}.$$

Dividing both sides of the above inequality by $f(L_2^{\frac{1}{\alpha}}x[g(t)])$, taking the lim sup of both sides as $t \to \infty$ and using (2.2), we obtain a contradiction to (2.1).
(II) Assume that $L_1x(t) < 0$, $L_2x(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Then we have

$$\int_s^t L_2x(u)\Delta u = L_1x(t) - L_1x(s) \leq -L_1x(s), \quad (t,s) \in [s,\infty)_{\mathbb{T}} \times [t_1,\infty)_{\mathbb{T}}.$$

Since L_2x is nonincreasing, we have

$$-L_1x(s) \geq (t-s)L_2x(t), \quad (t,s) \in [s,\infty)_{\mathbb{T}} \times [t_1,\infty)_{\mathbb{T}}$$

or

$$-x^{\Delta}(s) \geq a^{\frac{1}{\alpha}}(s)(t-s)^{\frac{1}{\alpha}} L_2^{\frac{1}{\alpha}}x(t), \quad (t,s) \in [s,\infty)_{\mathbb{T}} \times [t_1,\infty)_{\mathbb{T}}.$$

Integrating the above inequality from s to t we obtain

$$x(s) \geq B(t,s)L_2^{\frac{1}{\alpha}}x(t), \quad (t,s) \in [s,\infty)_{\mathbb{T}} \times [t_1,\infty)_{\mathbb{T}}, \tag{2.5}$$

where B is defined as in (1.7). Then there exists $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that

$$x[g(s)] \geq B(g(t),g(s))L_2^{\frac{1}{\alpha}}x[g(t)], \quad (g(t),g(s)) \in [g(s),\infty)_{\mathbb{T}} \times [t_2,\infty)_{\mathbb{T}}. \tag{2.6}$$

Integrating (1.1) from $g(t)$ to t and using (1.4) along with the above inequality we have

$$L_2x[g(t)] \geq \int_{g(t)}^t q(s)f(x[g(s)])\Delta s \geq f(L_2^{\frac{1}{\alpha}}x[g(t)]) \int_{g(t)}^t q(s)f(B(g(t),g(s)))\Delta s.$$

Dividing the above inequality by $f(L_2^{\frac{1}{\alpha}}x[g(t)])$, taking the lim sup of both sides of the above inequality as $t \to \infty$ and using (2.3), we obtain a contradiction to (2.1). $\square$

For the bounded solutions of (1.1) we have the following which is immediate from Theorem 2.1.

Corollary 2.2 *In addition to* (1.3) *and* (1.4)*, assume that* (2.1) *and* (2.3) *hold. Then all bounded solutions of* (1.1) *are oscillatory.*

Now we prove the following result.

Theorem 2.3 *Let* (1.3) *and* (1.4) *hold and assume that*

$$\lim_{u \to 0} \frac{u}{f(u^{\frac{1}{\alpha}})} = 0.$$

If

$$\limsup_{t \to \infty} \int_{g(t)}^t q(s)f(B(g(t),g(s)))\Delta s > 0,$$

where B is defined as in (1.7)*, then all bounded solutions of* (1.1) *are oscillatory.*

Proof Let x be a nonoscillatory bounded solution of (1.1). Without loss of generality assume that x is positive. Since x satisfies Case (II) in the proof of Theorem 2.1, we have (2.6). Integrating (1.1) from $g(t)$ to t and using (1.4) along with (2.6) yields

$$L_2x[g(t)] \geq \int_{g(t)}^{t} q(s)f(x[g(s)])\Delta s \geq f(L_2^{\frac{1}{\alpha}}x[g(t)]) \int_{g(t)}^{t} q(s)f(B(g(t),g(s)))\Delta s.$$

By dividing the above inequality by $f(L_2^{\frac{1}{\alpha}}x[g(t)])$ and taking the lim sup of both sides of the resulting inequality as $t \to \infty$, we obtain a contradiction. The proof is now complete. □

We now consider a special case of equation (1.1) of the form

$$\left(\frac{1}{a(t)}\left(x^{\Delta}(t)\right)^{\alpha}\right)^{\Delta\Delta} + q(t)x^{\beta}[g(t)] = 0, \tag{2.7}$$

where β is a ratio of two odd positive integers, and obtain some oscillatory criteria.

Theorem 2.4 *Let $\alpha \geq \beta$ and assume that*

$$\int^{\infty} q(s)A^{\beta}(g(s),t_0)\Delta s = \infty \tag{2.8}$$

and

$$\int^{\infty} q(s)B^{\beta}(g(s),t_0)\Delta s = \infty, \tag{2.9}$$

where A and B are defined as in (1.6) *and* (1.7)*, respectively. Then* (2.7) *is oscillatory.*

Proof Let x be a nonoscillatory solution of (2.7). Without loss of generality, assume that x is positive. We consider two cases as we did in the proof of Theorem 2.1.

(I) Assume that $L_1x(t) > 0$ and $L_2x(t) > 0$ for $t \geq t_1$. Using (2.4) in (2.7), we have

$$-L_2^{\Delta}x(t) = q(t)x^{\beta}[g(t)] \geq q(t)A^{\beta}(g(t),t_1)L_2^{\frac{\beta}{\alpha}}x[g(t)], \quad g(t) \in [t_2,\infty)_{\mathbb{T}}.$$

Integrating the above inequality from $g(t)$ to u and letting $u \to \infty$, we obtain

$$L_2^{1-\frac{\beta}{\alpha}}x[g(t)] \geq \int_{g(t)}^{\infty} q(s)A^{\beta}(g(s),t_1)\Delta s.$$

Since $\alpha \geq \beta$ and the right hand side of the above inequality is infinity by (2.8), we obtain a contradiction to the facts that L_2x is positive and nondecreasing.

(II) Assume that $L_1x(t) < 0$, $L_2x(t) > 0$ for $t \geq t_1$. Then we have (2.6). Using (2.6) in (2.7), for $(g(t), g(s)) \in [g(s), \infty)_{\mathbb{T}} \times [t_1, \infty)_{\mathbb{T}}$, we have

$$-L_2^{\Delta}x[g(t)] = q(t)x^{\beta}[g(t)] \geq q(t)B^{\beta}(g(t), g(s))L_2^{\frac{1}{\alpha}}x[g(t)].$$

Integrating the above inequality from $g(t)$ to u and letting $u \to \infty$, we obtain

$$L_2^{1-\frac{\beta}{\alpha}}x[g(t)] \geq \int_{g(t)}^{\infty} q(s)B^{\beta}(g(s), t_1)\Delta s.$$

Since $\alpha \geq \beta$ and the right hand side of the above inequality is infinity by (2.9), we obtain a contradiction to the facts that L_2x is positive and nondecreasing. $\square$

3 Oscillation Criteria for (1.2)

In this section, we investigate some oscillation criteria for solutions of the third-order delay equation (1.2).

Theorem 3.1 *Let* (1.3) - (1.5) *and* (2.1) *hold. Also assume that*

$$\frac{h(u^{\frac{1}{\alpha}})}{u} \geq b > 0 \tag{3.1}$$

for $u \neq 0$ and a constant b. If for $t \in [t_0, \infty)_{\mathbb{T}}$

$$\limsup_{t\to\infty} \int_{t}^{k(t)} q(s)h(A(k(s), k(t)))\Delta s > \frac{1}{b}, \tag{3.2}$$

and

$$\limsup_{t\to\infty} \int_{g(t)}^{t} q(s)f(B(g(t), t_0))\Delta s > \frac{1}{c}, \tag{3.3}$$

where A and B are defined as in (1.6) *and* (1.7)*, respectively, then* (1.2) *is oscillatory.*

Proof Let x be an eventually positive solution of (1.2). Then $L_3x(t) \geq 0$ eventually and so $L_1x(t)$ and $L_2x(t)$ are eventually of one sign. We now distinguish the following two cases:

(I) $L_1x(t) > 0$ and $L_2x(t) > 0$ eventually;

(II) $L_1x(t) > 0$ and $L_2x(t) < 0$ eventually.

(I) Assume that $L_1x(t) > 0$ and $L_2x(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Then we have

$$\int_{s}^{t} L_2x(\tau)\Delta\tau = L_1x(t) - L_1x(s) \leq L_1x(t), \quad (t, s) \in [s, \infty)_{\mathbb{T}} \times [t_1, \infty)_{\mathbb{T}}.$$

Since L_2x is nondecreasing, we have

$$L_1x(t) \geq (t - s)L_2x(s), \quad (t, s) \in [s, \infty)_{\mathbb{T}} \times [t_1, \infty)_{\mathbb{T}}.$$

This implies that

$$x^{\Delta}(t) \geq a^{\frac{1}{\alpha}}(t)(t-s)^{\frac{1}{\alpha}} L_2^{\frac{1}{\alpha}} x(s), \quad (t,s) \in [s,\infty)_{\mathbb{T}} \times [t_1,\infty)_{\mathbb{T}}.$$

Integrating both sides of the above inequality from s to t, we have

$$x(t) \geq \int_s^t a^{\frac{1}{\alpha}}(u)(u-s)^{\frac{1}{\alpha}} L_2^{\frac{1}{\alpha}} x(s) \Delta u, \quad (t,s) \in [s,\infty)_{\mathbb{T}} \times [t_1,\infty)_{\mathbb{T}}$$

or

$$x(t) \geq A(t,s) L_2^{\frac{1}{\alpha}} x(s), \quad (t,s) \in [s,\infty)_{\mathbb{T}} \times [t_1,\infty)_{\mathbb{T}},$$

where A is defined as in (1.6). Then

$$x[k(\tau)] \geq A(k(\tau),k(t)) L_2^{\frac{1}{\alpha}} x[k(t)], \quad (k(t),\tau,t) \in [\tau,\infty)_{\mathbb{T}} \times [t,\infty)_{\mathbb{T}} \times [t_1,\infty)_{\mathbb{T}}.$$

Using the above inequality and (1.5) in (1.2), we have

$$L_3 x(\tau) \geq p(\tau) h(x[k(\tau)]) \geq p(\tau) h(A(k(\tau),k(t))) h(L_2^{\frac{1}{\alpha}} x[k(t)]).$$

Integrating both sides of the above inequality from t to $k(t)$, we get

$$\frac{L_2 x[k(t)]}{h(L_2^{\frac{1}{\alpha}} x[k(t)])} \geq \int_t^{k(t)} p(\tau) h(A(k(\tau),k(t))) \Delta\tau.$$

Taking the $\limsup$ of both sides of the above inequality as $t \to \infty$, we obtain a contradiction to (3.1).

(II) Assume that $L_1 x(t) > 0$ and $L_2 x(t) < 0$ for $t \in [t_2,\infty)_{\mathbb{T}}$. Then we have

$$\int_s^t L_2 x(u) \Delta u = L_1 x(t) - L_1 x(s), \quad (t,s) \in [s,\infty)_{\mathbb{T}} \times [t_1,\infty)_{\mathbb{T}}.$$

Since $L_2 x$ is nondecreasing, we have

$$x^{\Delta}(s) \geq -a^{\frac{1}{\alpha}}(s)(t-s)^{\frac{1}{\alpha}} L_2^{\frac{1}{\alpha}} x(t), \quad (t,s) \in [s,\infty)_{\mathbb{T}} \times [t_1,\infty)_{\mathbb{T}}.$$

Integrating both sides of the above inequality from t_1 to t, we have

$$x(t) \geq -B(t,t_1) L_2^{\frac{1}{\alpha}} x(t), \quad t \in [t_1,\infty)_{\mathbb{T}},$$

where B is defined as in (1.7). This implies that there exists a $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that

$$x[g(t)] \geq -B(g(t),t_1) L_2^{\frac{1}{\alpha}} x[g(t)], \quad g(t) \in [t_1,\infty)_{\mathbb{T}}. \tag{3.4}$$

Using the above inequality and (1.4) in (1.2), we have

$$L_3 x(t) \geq q(t) f(x[g(t)]) \geq q(t) f(B(g(t),t_1)) f(-L_2^{\frac{1}{\alpha}} x[g(t)]).$$

Integrating both sides of the above inequality from $g(t)$ to t, we find

$$\frac{-L_2x[g(t)]}{f(-L_2^{\frac{1}{\alpha}}x[g(t)])} \geq \int_{g(t)}^{t} q(s)f(B(g(s),t_1))\Delta s.$$

Taking the lim sup of both sides of the above inequality as $t \to \infty$, we obtain a contradiction to (2.1). The proof is complete.

□

Theorem 3.2 *Assume that*

$$\frac{h^{\frac{1}{\alpha}}(u)}{u} \geq b_1 > 0, \tag{3.5}$$

and

$$\frac{f^{\frac{1}{\alpha}}(u)}{u} \geq c_1 > 0, \tag{3.6}$$

for $u \neq 0$ *and constants* b_1 *and* c_1. *If*

$$\limsup_{t\to\infty} \int_t^{k(t)} \left(a(s)\int_t^s \int_t^u p(r)\Delta r \Delta u\right)^{\frac{1}{\alpha}} \Delta s > \frac{1}{b_1} \tag{3.7}$$

and

$$\limsup_{t\to\infty} B(g(t),t_0)\left(\int_t^\infty q(s)\Delta s\right)^{\frac{1}{\alpha}} > \frac{1}{c_1}, \tag{3.8}$$

where B *is defined as in* (1.7), *respectively, then* (1.2) *is oscillatory.*

Proof Let x be an eventually positive solution of (1.2). As in the proof of Theorem 3.1 we have two cases to consider.

(I) Assume that $L_1x(t) > 0$ and $L_2x(t) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. Integrating (1.2) from t to s_1 yields

$$L_2x(s_1) \geq h(x[k(t)]) \int_t^{s_1} p(s)\Delta s.$$

Integrating the above inequality from t to $s_2 \in [s_1,\infty)_{\mathbb{T}}$ gives

$$x^{\Delta}(s_2) \geq h^{\frac{1}{\alpha}}(x[k(t)])\,[a(s_2)]^{\frac{1}{\alpha}}\left(\int_t^{s_2}\int_t^{s_1} p(s)\Delta s\Delta s_1\right)^{\frac{1}{\alpha}}.$$

Again integrating the above inequality from t to $k(t)$ we find

$$x(k(t)) \geq h^{\frac{1}{\alpha}}(x[k(t)])\int_t^{k(t)}\left(a(s_2)\int_t^{s_2}\int_t^{s_1} p(s)\Delta s\Delta s_1\right)^{\frac{1}{\alpha}}\Delta s_2.$$

Finally, dividing the above inequality by $h^{\frac{1}{\alpha}}(x(k(t)))$ and taking the lim sup of both sides of the above inequality as $t \to \infty$, we obtain a contradiction to (3.5).

(II) Assume that $L_1x(t) > 0$ and $L_2x(t) < 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Integrating

$$L_2^{\Delta}x(t) \geq q(t)f(x[g(t)])$$

from t to ∞ we have

$$-L_2x(t) \geq f(x[g(t)]) \int_t^{\infty} q(s)\Delta s.$$

Using (3.4) along with the above inequality, we have

$$x[g(t)] \geq B(g(t), t_1) \left(f(x[g(t)]) \int_t^{\infty} q(s)\Delta s \right)^{\frac{1}{\alpha}}.$$

Dividing the above inequality by $f^{\frac{1}{\alpha}}(x[g(t)])$ and taking the $\limsup$ of both sides of the resulting inequality as $t \to \infty$, we obtain a contradiction to (3.6). The proof is complete.

□

4 Examples

In this section we give examples to illustrate two of our main results. Recall

Theorem 4.1 *[1, Theorem 1.75] If $f \in C_{rd}$ and $t \in \mathbb{T}^{\kappa}$, then*

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t),$$

and

Theorem 4.2 *[1, Theorem 1.79 (ii)] If $[a, b]$ consists of only isolated points and $a < b$, then*

$$\int_a^b f(t)\Delta t = \sum_{t \in [a,b)} \mu(t)f(t).$$

Our first example illustrates Theorem 2.1.

Example 4.3 *Consider the third-order delay dynamic equation*

$$\left(\frac{1}{a(t)} \left(x^{\Delta}(t)\right)^{\alpha} \right)^{\Delta\Delta} + q(t)x^{\alpha}\left(\frac{t}{3}\right) = 0 \tag{4.1}$$

where $t \in \mathbb{T} = 3^{\mathbb{N}_0}$. Here $\alpha = \frac{1}{3}$, $q(t) = 3 + (-1)^{\frac{\ln t}{\ln 3}}$, $g(t) = \frac{t}{3}$, and $a(u) = f(u) = u^{\alpha}$. Observe that if $t \in \mathbb{T}$, then

$$q(t) = 3 + (-1)^n = \begin{cases} 2, & n \text{ odd} \\ 4, & n \text{ even}. \end{cases}$$

First we show that (1.3) holds. If $s = 3^m$ and $t = 3^n$, $m, n \in \mathbb{N}_0$, we have

$$\int_1^\infty a^{\frac{1}{\alpha}}(s)\Delta s = \lim_{t\to\infty}\int_1^t a^{\frac{1}{\alpha}}(s)\Delta s = \lim_{n\to\infty}\sum_{s=1}^{\rho(3^n)} s(3s-s) = \lim_{n\to\infty}\sum_{s=1}^{3^{n-1}} 2s^2 = \infty.$$

It is clear that f belongs to $C(\mathbb{R},\mathbb{R})$, is nondecreasing for $x \neq 0$, and satisfies $xf(x) > 0$ for $x \neq 0$ and (1.4). Also, (2.1) holds since

$$\frac{f^{\frac{1}{\alpha}}(u)}{u} = \frac{u}{u} = 1 = c > 0, \quad u \neq 0.$$

Observe that if $u = 3^k$ and $s = 3^m$ for $k, m \in \mathbb{N}_0$, then

$$A(g(s),1) = 2\sum_{u=1}^{\rho(3^{m-1})} u^2(u-1)^3 = 2\sum_{k=0}^{m-2} 3^{2k}(3^k-1)^3 = 2\sum_{k=1}^{m-2} 3^{2k}(3^k-1)^3.$$

Note $3^k - 1 > 1$ for $k \in \mathbb{N}$. Hence

$$A(g(s),1) > 2\sum_{k=1}^{m-2} 3^{2k} > 2\cdot 3^{2(m-2)},$$

and since f is nondecreasing and $q(t) \geq 2$ on $\mathbb{T}$, we obtain

$$q(s)f((A(g(s),1))) \geq 2^{\frac{4}{3}}3^{\frac{2}{3}(m-2)}.$$

It follows that if $t \in [1,\infty)_{\mathbb{T}}$

$$\begin{aligned}
\limsup_{t\to\infty}\int_{g(t)}^t q(s)f(A(g(s),t_0))\Delta s &\geq \limsup_{n\to\infty}\sum_{s=\frac{t}{3}}^{3^{n-1}} 2^{\frac{4}{3}}3^{\frac{2}{3}(n-2)}2s \\
&= 2^{\frac{7}{3}}\limsup_{n\to\infty}\sum_{m=n-1}^{n-1} 3^{\frac{2}{3}(n-2)}3^m \\
&> 1,
\end{aligned}$$

and so (2.2) holds. It remains to show that (2.3) holds for $t \in [1,\infty)_{\mathbb{T}}$. This requires that we determine $B(g(t),g(s))$. Using the above representations of u, s, t, we have

$$B(g(t),g(s)) = 2\sum_{k=m-1}^{n-2} 3^{2k}\left(3^{n-1}-3^k\right)^3 > 2\cdot 3^{2(n-2)}\left(3^{n-1}-3^{n-2}\right)^3.$$

The monotonicity of f and the fact that $q(t) \geq 2$ on $\mathbb{T}$ yield

$$q(s)f(B(g(t),g(s))) > 2^{\frac{4}{3}}\cdot 3^{\frac{2}{3}(n-2)}\left(3^{n-1}-3^{n-2}\right) = 2^{\frac{7}{3}}\cdot 3^{\frac{5}{3}(n-2)}.$$

Therefore

$$
\begin{aligned}
\limsup_{t\to\infty}\int_{\frac{t}{3}}^{t} q(s)f(B(g(t),g(s)))\Delta s &> \limsup_{n\to\infty} 2^{\frac{7}{3}}\cdot 3^{\frac{5}{3}(n-2)}\sum_{s=3^{n-1}}^{3^{n-1}}\mu(s)\\
&= \limsup_{n\to\infty} 2^{\frac{10}{3}}\cdot 3^{\frac{5}{3}(n-2)}\sum_{m=n-1}^{n-1} 3^{m}\\
&> 1,
\end{aligned}
$$

which shows that (2.3) *holds. By Theorem 2.1,* (4.1) *is oscillatory.*

Our second example illustrates Theorem 3.2.

Example 4.4 *Consider the third-order advanced dynamic equation*

$$
\left(\frac{1}{a(t)}\left(x^{\Delta}(t)\right)^{\alpha}\right)^{\Delta\Delta} = q(t)x^{\alpha}\left(\frac{t}{q}\right) + p(t)x^{\alpha}(q^{3}t), \tag{4.2}
$$

where α *is the ratio of two positive odd integers and* $t \in \mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$. *Here* $a(t) = \left(\frac{q}{(q-1)t}\right)^{\alpha}$, $q(t) = \frac{\frac{q}{q-1}+1+(-1)^{\frac{\ln t}{\ln q}}}{(q-1)t}$, $p(t) = \frac{1}{t^{2}q^{\alpha}(q-1)^{2}}$, $g(t) = \frac{t}{q}$, $k(t) = q^{3}t$, *and* $h(u) = f(u) = u^{\alpha}$. *Then*

$$
\frac{h^{\frac{1}{\alpha}}(u)}{u} = \frac{f^{\frac{1}{\alpha}}(u)}{u} = \frac{u}{u} = 1 = b_1 = c_1 > 0, \quad u \neq 0,
$$

and so (3.5) and (3.6) hold. Next we show that (3.7) holds. For $t \in [1,\infty)_{\mathbb{T}}$, we have

$$
\begin{aligned}
&\limsup_{t\to\infty} \int_t^{k(t)} \left(a(s) \int_t^s \int_t^u p(r)\Delta r \Delta u \right)^{\frac{1}{\alpha}} \Delta s \\
&= \limsup_{t\to\infty} \left[\int_t^{\sigma(t)} \left(a(s) \int_t^s \int_t^u p(r)\Delta r \Delta u \right)^{\frac{1}{\alpha}} \Delta s + \int_{\sigma(t)}^{\sigma^2(t)} \left(a(s) \int_t^s \int_t^u p(r)\Delta r \Delta u \right)^{\frac{1}{\alpha}} \Delta s \right. \\
&\qquad \left. + \int_{\sigma^2(t)}^{\sigma^3(t)} \left(a(s) \int_t^s \int_t^u p(r)\Delta r \Delta u \right)^{\frac{1}{\alpha}} \Delta s \right] \\
&= \limsup_{t\to\infty} \left[\mu(t) \left(a(t) \int_t^t \int_t^u p(r)\Delta r \Delta u \right)^{\frac{1}{\alpha}} + \mu(\sigma(t)) \left(a(\sigma(t)) \int_t^{\sigma(t)} \int_t^u p(r)\Delta r \Delta u \right)^{\frac{1}{\alpha}} \right. \\
&\qquad \left. + \mu(\sigma^2(t)) \left(a(\sigma^2(t)) \int_t^{\sigma^2(t)} \int_t^u p(r)\Delta r \Delta u \right)^{\frac{1}{\alpha}} \right] \\
&= \limsup_{t\to\infty} \left[\mu(\sigma(t)) \left(a(\sigma(t))\mu(t) \int_t^t p(r)\Delta r \right)^{\frac{1}{\alpha}} \right. \\
&\qquad \left. + \mu(\sigma^2(t)) \left(a(\sigma^2(t)) \int_t^{\sigma^2(t)} \int_t^u p(r)\Delta r \Delta u \right)^{\frac{1}{\alpha}} \right] \\
&= \limsup_{t\to\infty} \left[\mu(\sigma^2(t)) \left(a(\sigma^2(t)) \right)^{\frac{1}{\alpha}} \left(\mu(t) \int_t^t p(r)\Delta r + \mu(\sigma(t)) \int_t^{\sigma(t)} p(r)\Delta r \right)^{\frac{1}{\alpha}} \right] \\
&= \limsup_{t\to\infty} \left[\mu(\sigma^2(t)) \left(a(\sigma^2(t)) \right)^{\frac{1}{\alpha}} \left(\mu(\sigma(t))\mu(t)p(t) \right)^{\frac{1}{\alpha}} \right] \\
&= \limsup_{t\to\infty} \left[(q-1)q^2 t \,(q-1)^{\frac{2}{\alpha}} \left(qt^2 \right)^{\frac{1}{\alpha}} \frac{q}{(q-1)q^2 t} \frac{1}{t^{\frac{2}{\alpha}} q (q-1)^{\frac{2}{\alpha}}} \right] \\
&= \limsup_{t\to\infty} \left[q^{\frac{1}{\alpha}} \right] \\
&= q^{\frac{1}{\alpha}} > 1.
\end{aligned}
$$

Since $q(t) > 0$ for all $t \in \mathbb{T}$, we have

$$
\int_t^\infty q(s)\Delta s \ge \int_t^{\sigma(t)} q(s)\Delta s = \mu(t)q(t) = \frac{q}{q-1} + 1 + (-1)^{\frac{\ln t}{\ln q}} > \frac{q}{q-1}.
$$

This implies

$$
\begin{aligned}
&\limsup_{t\to\infty} B(g(t),t_0) \left(\int_t^\infty q(s)\Delta s \right)^{\frac{1}{\alpha}} \\
&\ge \left(\frac{q}{q-1} \right)^{\frac{1}{\alpha}} \limsup_{t\to\infty} \int_1^{\frac{t}{q}} a^{\frac{1}{\alpha}}(u) \left(\frac{t}{q} - u \right)^{\frac{1}{\alpha}} \Delta u \\
&= \left(\frac{q}{q-1} \right)^{\frac{1}{\alpha}} \limsup_{n\to\infty} \sum_{k=0}^{n-2} \frac{q}{(q-1)q^k} \left(\frac{q^n}{q} - q^k \right)^{\frac{1}{\alpha}} q^k (q-1) \\
&= q \left(\frac{q}{q-1} \right)^{\frac{1}{\alpha}} \limsup_{n\to\infty} \sum_{k=0}^{n-2} \left(\frac{q^n}{q} - q^k \right)^{\frac{1}{\alpha}} \\
&> 1.
\end{aligned}
$$

Thus (3.8) *holds. By Theorem 3.2,* (4.2) *is oscillatory.*

5 Discussion

While we were able to unify most results for (1.1) given in [3] and [4], the comparison result

Theorem Let (1.3) – (1.4) hold. If the first-order delay dynamic equations

$$y^{\Delta}(t) + q(t)f\left(A(g(t),t_0)\right) f\left(y^{\frac{1}{\alpha}}[g(t)]\right) = 0 \tag{5.1}$$

and

$$z^{\Delta}(t) + q(t)f\left(B\left(\frac{t+g(t)}{2}, g(t)\right)\right) f\left(z^{\frac{1}{\alpha}}\left[\frac{t+g(t)}{2}\right]\right) = 0 \tag{5.2}$$

are oscillatory, then (1.1) is oscillatory.

cannot be extended since $\dfrac{t+g(t)}{2} \in \mathbb{T}$ is satisfied for few time scales. While the result holds for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, this condition is not satisfied on $q^{\mathbb{N}},\ q > 1$. Being aware that time scales are not generally closed under addition, we were able to prove the following:

Theorem 5.1 *Let* (1.3) – (1.4) *hold. Furthermore, assume the delay function* $g : \mathbb{T} \to \mathbb{T}$ *is a bijection. If the first-order delay dynamic equations*

$$y^{\Delta}(t) + q(t)f\left(A(g(t),t_0)\right) f\left(y^{\frac{1}{\alpha}}[g(t)]\right) = 0 \tag{5.3}$$

and

$$z^{\Delta}(t) + q(t)f\left(B\left(\frac{t+g(t)}{2}, g(t)\right)\right) f\left(z^{\frac{1}{\alpha}}\left[\frac{t+g(t)}{2}\right]\right) = 0 \tag{5.4}$$

are oscillatory, where $(t+g(t))/2 \in \mathbb{T}$, *then* (1.1) *is oscillatory.*

In order to prove Theorem 5.1, it is necessary to define the function $F : [t_0,\infty)_{\mathbb{T}} \times [0,\infty) \mapsto [0,\infty)$ to be a nondecreasing function with respect to its second argument and with the property that $F(.,z(.)) \in \mathrm{C_{rd}}\left([t_0,\infty)_{\mathbb{T}},[0,\infty)\right)$ for any function $z \in \mathrm{C_{rd}}([t_0,\infty)_{\mathbb{T}},[0,\infty))$. It is also necessary to assume that $\tau : \mathbb{T} \to \mathbb{T}$ is a bijection with $\tau(t) < t$ for all $t \in \mathbb{T}$ and $\lim_{t\to\infty} \tau(t) = \infty$, and to use the following definition and theorem.

Definition 5.2 *Let* $t_1 \in [t_0,\infty)_{\mathbb{T}}$. *By a solution of the dynamic inequality*

$$y^{\Delta}(t) + F\left(t, y\left[\tau(t)\right]\right) \leq 0 \tag{5.5}$$

on an interval $[t_1,\infty)_{\mathbb{T}}$, *we mean a rd-continuous function* y *defined on the interval* $[\tau(t_1),\infty)_{\mathbb{T}}$, *which is rd-continuously differentiable on* $[t_1,\infty)_{\mathbb{T}}$ *and satisfies* (5.5) *for all* $t \in [t_1,\infty)_{\mathbb{T}}$. *A solution* y *of* (5.5) *is said to be positive if* $y(t) > 0$ *for every* $t \in [\tau(t_1),\infty)_{\mathbb{T}}$.

Theorem 5.3 *Let y be a positive solution on an interval $[t_1,\infty)_{\mathbb{T}}$, $t_1 \geq t_0$ of the delay dynamic inequality (5.5). Moreover, we assume that F is positive on any set of the form $[\widehat{t},\tau^{-1}(\widehat{t})]_{\mathbb{T}} \times (0,\infty)$, $\widehat{t} \in [t_1,\infty)_{\mathbb{T}}$. Then there exists a positive solution x on $[\tau^{-1}(t_1),\infty)_{\mathbb{T}}$ of the delay dynamic equation*

$$x^{\Delta}(t) + F(t, x[\tau(t)]) = 0 \tag{5.6}$$

such that

$$\lim_{t\to\infty} x(t) = 0$$

and

$$x(t) \leq y(t) \text{ for every } t \in [t_1,\infty)_{\mathbb{T}}.$$

Proof Let y be a positive solution (5.5). From (5.5) we obtain for all $\widetilde{t}$, $t \in \mathbb{T}$ with $\widetilde{t} \geq t \geq t_1$

$$y(t) \geq y(\tilde{t}) + \int_t^{\widetilde{t}} F(s, y[\tau(s)])\Delta s > \int_t^{\widetilde{t}} F(s, y(\tau(s)))\Delta s. \tag{5.7}$$

Hence, letting $\widetilde{t} \to \infty$ we get

$$y(t) \geq \int_t^{\infty} F(s, y[\tau(s)])\Delta s \text{ for every } t \in [t_1,\infty)_{\mathbb{T}}. \tag{5.8}$$

Let X be the set of all nonnegative continuous functions x on the interval $[t_1,\infty)_{\mathbb{T}}$ with $x(t) \leq y(t)$ for every $t \geq t_1$. Then by using (5.8) we can easily verify that for any function x in X the formula

$$(Sx)(t) = \begin{cases} \int\limits_t^{\infty} F(s, x[\tau(s)])\Delta s, & t \in [\tau^{-1}(t_1),\infty)_{\mathbb{T}} \\ \int\limits_{\tau^{-1}(t_1)}^{\infty} F(s, x[\tau(s)])\Delta s + \int\limits_t^{\tau^{-1}(t_1)} F(s, y[\tau(s)])\Delta s, & t \in [t_1, \tau^{-1}(t_1))_{\mathbb{T}} \end{cases}$$

defines an operator $S : X \to X$. If $x_1, x_2 \in X$ and $x_1(t) \leq x_2(t)$ for $t \in [t_1,\infty)_{\mathbb{T}}$, then we also have $(Sx_1)(t) \leq (Sx_2)(t)$ for $t \in [t_1,\infty)_{\mathbb{T}}$ since F is nondecreasing with respect to its second argument. Thus, the operator S is monotone. Next, we put

$$x_0 = y, \quad \text{and } x_\nu = Sx_{\nu-1}, \ \nu = 1, 2, ...$$

and observe that $\{x_\nu\}_{\nu=0,1,2}$ is a decreasing sequence of functions in X. Furthermore, define

$$x = \lim_{\nu\to\infty} x_\nu \text{ pointwise on } [t_1,\infty)_{\mathbb{T}}.$$

Then by applying the Lebesgue dominated convergence theorem, we obtain $x = Sx$. That is

$$x(t) = \begin{cases} \int\limits_{t}^{\infty} F(s, x[\tau(s)])\,\Delta s & \text{if } t \in [\tau^{-1}(t_1), \infty)_{\mathbb{T}} \\ \int\limits_{\tau^{-1}(t_1)}^{\infty} F(s, x[\tau(s)])\,\Delta s + \int\limits_{t}^{\tau^{-1}(t_1)} F(s, y[\tau(s)])\,\Delta s & \text{if } t \in [t_1, \tau^{-1}(t_1))_{\mathbb{T}}. \end{cases} \tag{5.9}$$

From (5.9) it follows that

$$x^{\Delta}(t) + F(t, x[\tau(t)]) = 0 \text{ for all } t \in [\tau^{-1}(t_1), \infty)_{\mathbb{T}}$$

and hence x is a solution of (5.6) on the interval $t \in [\tau^{-1}(t_1), \infty)_{\mathbb{T}}$. Also (5.9) yields

$$\lim_{t\to\infty} x(t) = 0.$$

Moreover, it is clear that $x(t) \le y(t)$ for $t \in [t_1, \infty)_{\mathbb{T}}$. It remains to prove that x is positive on the interval $[t_1, \infty)_{\mathbb{T}}$. Taking into account that y is positive on the interval $[\tau(t_1), \infty)_{\mathbb{T}}$ and the positivity of F on $[t_1, \tau^{-1}(t_1)]_{\mathbb{T}} \times (0, \infty)$ we have

$$x(t) \ge \int_{t}^{\tau^{-1}(t_1)} F(s, y[\tau(s)])\,\Delta s > 0$$

for each $t \in [t_1, \tau^{-1}(t_1))_{\mathbb{T}}$. So, x is positive on an interval $[t_1, \tau^{-1}(t_1))_{\mathbb{T}}$. Next we will show that x is also positive on $[\tau^{-1}(t_1), \infty)_{\mathbb{T}}$. Assume that $\widehat{t}$ is the first zero of x in $[\tau^{-1}(t_1), \infty)_{\mathbb{T}}$. Then $x(t) > 0$ for $t \in [\tau^{-1}(t_1), \widehat{t})_{\mathbb{T}}$, and $x\left(\widehat{t}\right) = 0$. Then (5.9) yields

$$0 = x\left(\widehat{t}\right) = \int_{\widehat{t}}^{\infty} F(s, x[\tau(s)])\,\Delta s$$

and consequently

$$F(s, x[\tau(s)]) = 0 \text{ for all } s \in [\widehat{t}, \infty)_{\mathbb{T}}.$$

That is, we can choose a $t^* \in [\widehat{t}, \tau^{-1}\left(\widehat{t}\right))_{\mathbb{T}}$ such that

$$F(t^*, x[\tau(t^*)]) = 0.$$

On the other hand, taking into account that $x(t) > 0$ for $t \in [\tau^{-1}(t_1), \widehat{t})_{\mathbb{T}}$ and the positivity of F on $[\widehat{t}, \tau^{-1}\left(\widehat{t}\right)]_{\mathbb{T}} \times (0, \infty)$ we get

$$F(t^*, x[\tau(t^*)]) > 0.$$

This leads to a contradiction and the proof is complete. □

Note that Theorem 5.3 holds for any unbounded time scale $\mathbb{T}$. Now we present the proof of Theorem 5.1.

Proof Let x be an eventually positive solution of (1.1). We consider two cases as we did in the proof of Theorem 2.1.

(I) Assume that $L_1x(t) > 0$ and $L_2x(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Then (2.4) holds. Using this and (1.4) in (1.1), we obtain

$$-L_2^{\Delta}x(t) = q(t)f(x[g(t)]) \geq q(t)f(A(g(t),t_1))f(L_2^{\frac{1}{\alpha}}x[g(t)]), \quad t \in [t_2,\infty)_{\mathbb{T}}$$

or

$$y^{\Delta}(t) + q(t)f(A(g(t),t_1))f(y^{\frac{1}{\alpha}}[g(t)]) \leq 0, \quad t \in [t_2,\infty)_{\mathbb{T}},$$

where $y(t) := L_2x(t)$ for $t \in [t_2,\infty)_{\mathbb{T}}$. Since $L_2x(t) > 0$ for all $t \in [t_2,\infty)$ and $A(g(t),t_1) > 0$ for all $t \in [t_1,\infty)$, by Theorem 5.3, there exists a positive solution z of (5.3) such that $\lim_{t\to\infty} z(t) = 0$, which contradicts the hypotheses that (5.3) is oscillatory.

(II) Assume that $L_1x(t) < 0$ and $L_2x(t) > 0$ for $t \geq t_1$. As in the proof of Theorem 2.1, we have (2.5). Then for $t \in [t_1,\infty)_{\mathbb{T}}$, we have

$$x[g(t)] \geq B\left(\frac{t+g(t)}{2}, g(t)\right) L_2^{\frac{1}{\alpha}}x\left(\frac{t+g(t)}{2}\right).$$

Using this inequality and (1.4) in (1.1) for $t \in [t_1,\infty)_{\mathbb{T}}$ yields

$$-L_2^{\Delta}x(t) = q(t)f(x[g(t)]) \geq q(t)f\left(B\left(\frac{t+g(t)}{2}, g(t)\right)\right) f\left(L_2^{\frac{1}{\alpha}}x\left[\frac{t+g(t)}{2}\right]\right)$$

or

$$-z^{\Delta}(t) \geq q(t)f\left(B\left(\frac{t+g(t)}{2}, g(t)\right)\right) f\left(z^{\frac{1}{\alpha}}\left[\frac{t+g(t)}{2}\right]\right),$$

where $z(t) := L_2x(t)$ for $t \in [t_1,\infty)_{\mathbb{T}}$. Similar to Case (I) above, by Theorem 5.3, there exists a positive solution y of (5.4) such that $\lim_{t\to\infty} y(t) = 0$, which contradicts the fact that (5.4) is oscillatory.

□

In [3] and [4], the authors prove a comparison result for (1.2) similar to the one given at the beginning of this section. That result involved $\frac{t+k(t)}{2}$. Again, since time scales are not generally closed under addition, this result cannot be extended to a general time scale $\mathbb{T}$.

Competing interests
The authors declare that they have no competing interests.

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