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# A GENERALIZED UPPER AND LOWER METHOD FOR SINGULAR BOUNDARY VALUE PROBLEMS FOR QUASILINEAR DYNAMIC EQUATIONS

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ABSTRACT. In this paper, we obtain some existence results for a singular boundary value problem (BVP) for quasilinear dynamic equations on time scales. In particular, our nonlinearity may be singular in its dependent variable and is allowed to change sign.

## 1. INTRODUCTION

In this paper, we consider the singular BVP

$$\text{(1)} \qquad \begin{cases} \left(\Phi(y^{\Delta}(t))\right)^{\Delta} + q(t)f(t, y^{\sigma}(t)) = 0, \quad t \in [a, b] \\ y(a) = 0 = y(\sigma^2(b)), \end{cases}$$

on a time scale ($\mathbb{T}$) which is a nonempty closed subset of real numbers. Throughout this paper we assume that $\Phi(s) = |s|^{p-2}s,\ s > 1,\ f(t, x) : [a, b] \times (0, \infty) \to \mathbb{R}$ is continuous, and $f(t, x)$ may be singular or may change sign.

We let $\mathrm{C}([a, \sigma^2(b)])$ and $\mathrm{C}^1([a, \sigma(b)])$ be the classes of maps $y$ continuous on $[a, \sigma^2(b)]$ and continuously differentiable on $[a, \sigma(b)]$, respectively. And $||y|| = \max |y(t)|$ for $t \in [a, \sigma^2(b)]$. By a solution $y$ of BVP (1) we mean a function $y \in \mathrm{C}([a, \sigma^2(b)], \mathbb{R}) \cap \mathrm{C}^1([a, \sigma(b)], \mathbb{R})$, $\Phi(y^{\Delta}) \in \mathrm{C}^1([a, b], \mathbb{R})$, and $y$ satisfies BVP (1).

We note that (1) is a dynamic model of the p-Laplacian equation which occurs in the study of many diffusion phenomena. We refer the readers [6], and [7] for continuous case and [1], [8], [9], and [10] for discrete case.

The paper is organized as follows: In Section 2, we briefly introduce the theory of time scales. In Section 3, we prove some comparison results and consider the following BVP

$$\text{(2)} \qquad \begin{cases} (\Phi(y^{\Delta}(t)))^{\Delta} + F(t, y^{\sigma}(t)) = 0, \quad t \in [a, b] \\ y(a) = A, \quad y(\sigma^2(b)) = B, \end{cases}$$

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where $A$ and $B$ are given real numbers, and $F(t,x) : [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous in $x$. Finally, we obtain some existence results for solutions of the singular BVP (1) in the last section, where $\mathbb{T}$ contains isolated points.

## 2. Time Scale Calculus

In this section, we introduce a calculus on time scales including preliminary results. An introduction with applications and advances in dynamic equations are given in [4, 5]. We define the interval $[a,b] := \{t \in \mathbb{T} : [a,b] \cap \mathbb{T}\}$. There are two jump operators, namely the *forward jump operator* $\sigma$ and the *backward jump operator* $\rho$. We define them as follows:

$$\sigma(t) := \inf\{s > t : s \in \mathbb{T}\} \in \mathbb{T}, \quad \rho(t) := \sup\{s < t : s \in \mathbb{T}\} \in \mathbb{T}$$

for all $t \in \mathbb{T}$, where we put $\inf(\emptyset) = \sup \mathbb{T}$, $\sup(\emptyset) = \inf \mathbb{T}$. If $\sigma(t) > t$, we say t is *right-scattered*, while if $\rho(t) < t$, we say $t$ is *left-scattered.* If $\sigma(t) = t$, we say $t$ is *right-dense*, while if $\rho(t) = t$, we say $t$ is *left-dense.* The *graininess function* $\mu : \mathbb{T} \mapsto [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

Assume $f : \mathbb{T} \mapsto \mathbb{R}$ and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood $U$ of $t$ such that

$$\big|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\big| \le \epsilon\, |\sigma(t) - s|$$

for all $s \in U$. We call $f^\Delta(t)$ the *delta derivative* of $f(t)$ at $t$, and it turns out that $^\Delta$ is the usual derivative if $\mathbb{T} = \mathbb{R}$ and the usual forward difference operator $\Delta$ if $\mathbb{T} = \mathbb{Z}$. We have the following two formulas:

(i) $f^\Delta(t) = \dfrac{f(\sigma(t)) - f(t)}{\mu(t)}$ for right-scattered points in $\mathbb{T}$;

(ii) $f^\Delta(t) = \lim\limits_{s \to t} \dfrac{f(t) - f(s)}{t - s}$ for right-dense points in $\mathbb{T}$ if the limit exists.

We have another useful formula for $f^\Delta$, which is valid for any point in $\mathbb{T}$.

$$f^\sigma(t) = f(t) + \mu(t) f^\Delta(t), \quad \text{where} \quad f^\sigma = f \circ \sigma. \tag{3}$$

If $f, g : \mathbb{T} \mapsto \mathbb{R}$ are differentiable at $t \in \mathbb{T}^\kappa$, then the product and quotient rules are as follows:

$$(fg)^\Delta(t) = f^\Delta(t) g(t) + f^\sigma(t) g^\Delta(t)$$

and

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t) g(t) - f(t) g^\Delta(t)}{g(t) g^\sigma(t)} \quad \text{if} \quad g(t) g^\sigma(t) \neq 0.$$

We say $f : \mathbb{T} \mapsto \mathbb{R}$ is *rd-continuous* provided $f$ is continuous at each right-dense point $t \in \mathbb{T}$ and whenever $t \in \mathbb{T}$ is left-dense $\lim_{s \to t^-} f(s)$ exists as a finite number.

A function $F : \mathbb{T}^\kappa \mapsto \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \mapsto \mathbb{R}$ provided $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$. Every rd-continuous function has an antiderivative. In this case we define the integral of $f$ by

$$\int_a^t f(s)\Delta s = F(t) - F(a) \quad \text{for} \quad t \in \mathbb{T}.$$

The proof of following result can also be found in [4].

**Theorem 2.1.** *The solution of the BVP*

$$\left\{ \begin{array}{c} y^{\Delta\Delta}(t) + u^\sigma(t) = 0, \quad t \in [a,b] \\ y(a) = 0 = y(\sigma^2(b)), \end{array} \right. \tag{4}$$

*is given by*

$$y(t) = -\int_a^{\sigma(b)} G(t,s) u^\sigma(s) \Delta s,$$

*where $G(t,s)$ is the Green function of the BVP*

$$\left\{ \begin{array}{c} x^{\Delta\Delta}(t) = 0, \quad t \in [a,b] \\ x(a) = 0 = x(\sigma^2(b)), \end{array} \right.$$

*and*

$$G(t,s) = \left\{ \begin{array}{ll} \dfrac{-(t-a)(\sigma^2(b) - \sigma(s))}{\sigma^2(b) - a}, & \text{if } t \leq s, \\ -\dfrac{(\sigma(s) - a)(\sigma^2(b) - t)}{\sigma^2(b) - a}, & \text{if } t \geq s. \end{array} \right.$$

## 3. Comparison Principles

In this section, we obtain existence results for BVP (2). These necessary results are to get the existence of solutions for BVP (1). The discrete version of the following lemma can be found in [2].

**Lemma 3.1.** *Let $u \in C([a, \sigma^2(b)], \mathbb{R})$ satisfying $u(t) \geq 0$ on $[a, \sigma^2(b)]$. If $y \in C([a, \sigma^2(b)], \mathbb{R})$ satisfies BVP* (4), *then*

$$y(t) \geq r(t)||y|| \quad \text{for} \quad t \in [a, \sigma^2(b)], \tag{5}$$

*where*

$$r(t) = \min\left\{ \frac{t-a}{\sigma^2(b) - a}, \frac{\sigma^2(b) - t}{\sigma^2(b) - a} \right\}. \tag{6}$$

*Proof.* Inequality (5) is true when $t = a$ or $t = \sigma^2(b)$. If $||y|| = 0$, then (5) holds. Therefore, it is enough to consider (5) when $||y|| \neq 0$, $t \neq a$, $t \neq \sigma^2(b)$ since $r(a) = r(\sigma^2(b)) = 0$. By Theorem (2.1), we obtain

$$y(t) = -\int_a^{\sigma(b)} G(t,s)u^\sigma(s)\Delta s, \quad t \in [a, \sigma^2(b)], \; s \in [a, \sigma(b)].$$

Note that $G(t,s) < 0$ for $t \in [a, \sigma^2(b)]$, $s \in [a, \sigma(b)]$. This implies that $y(t) \geq 0$ for $t \in [a, \sigma^2(b)]$. Let $||y|| = y(t_0)$ and consider

$$y(t) = -\int_a^{\sigma(b)} \frac{G(t,s)G(t_0,s)}{G(t_0,s)} u^\sigma(s)\Delta s, \quad t_0 \in [a, \sigma^2(b)]. \tag{7}$$

We now show that

$$\frac{G(t,s)}{G(t_0,s)} \geq r(t) \tag{8}$$

for $t \in [a, \sigma^2(b)]$, $s \in [a, \sigma(b)]$. When $t > t_0$, we consider three cases: If $a \leq s < t_0$, then

$$\frac{G(t,s)}{G(t_0,s)} = \frac{\sigma^2(b) - t}{\sigma^2(b) - t_0} \geq \frac{\sigma^2(b) - t}{\sigma^2(b) - a} \geq r(t).$$

If $t_0 \leq s < t$, then

$$\frac{G(t,s)}{G(t_0,s)} = \frac{(\sigma(s) - a)(\sigma^2(b) - t)}{(t_0 - a)(\sigma^2(b) - \sigma(s))} \geq \frac{\sigma^2(b) - t}{\sigma^2(b) - \sigma(s)} \geq \frac{\sigma^2(b) - t}{\sigma^2(b) - a} \geq r(t),$$

since $\sigma(s) \geq t_0$. If $t \leq s \leq \sigma(b)$, then

$$\frac{G(t,s)}{G(t_0,s)} = \frac{t - a}{t_0 - a} \geq \frac{t - a}{\sigma^2(b) - a} \geq r(t).$$

When $t \leq t_0$, (8) can be shown similarly. Therefore,

$$y(t) \geq r(t)\left[-\int_a^{\sigma(b)} G(t_0,s)u^\sigma(s)\Delta s\right] = r(t)y(t_0) = r(t)||y||,$$

and so this completes the proof. □

Next we present three more lemmas whose discrete versions can be found in [8].

**Lemma 3.2.** *If* $y \in C([a, \sigma^2(b)], \mathbb{R})$ *satisfies*

$$\begin{cases} y^{\Delta\Delta}(t) \leq 0, & t \in [a,b] \\ y(a) \geq 0, & y(\sigma^2(b)) \geq 0, \end{cases}$$

*then* $y(t) \geq 0$ *on* $[a, \sigma^2(b)]$.

*Proof.* Set $Q(t) = y(a) + \dfrac{y(\sigma^2(b)) - y(a)}{\sigma^2(b) - a}(t-a)$, $t \in [a, \sigma^2(b)]$. Let $y(t) = v(t) + Q(t)$. Then $v^{\Delta}(t) = y^{\Delta}(t) - \dfrac{y(\sigma^2(b)) - y(a)}{\sigma^2(b) - a}$. This implies that $v^{\Delta\Delta}(t) = y^{\Delta\Delta}(t) \le 0$, $t \in [a, b]$ and $v(a) = v(\sigma^2(b)) = 0$. By Lemma 3.1, $v(t) \ge r(t)||v|| \ge 0$ on $[a, \sigma^2(b)]$ and so $y(t) \ge Q(t)$ on $[a, \sigma^2(b)]$. Since $y(a) \ge 0$ and $y(\sigma^2(b)) \ge 0$,

$$Q(t) = y(a) - y(a)\frac{t-a}{\sigma^2(b) - a} + \frac{y(\sigma^2(b))}{\sigma^2(b) - a} = y(a)\frac{\sigma^2(b) - t}{\sigma^2(b) - a} + \frac{y(\sigma^2(b))}{\sigma^2(b) - a} \ge 0$$

on $[a, \sigma^2(b)]$ and so this completes the proof. □

*Remark* 3.1. Lemma 3.2 can be obtained from the following lemma. However, the proof is longer in this case.

The following lemma is known as "a strong comparison principle" in the literature.

**Lemma 3.3.** *If* $y \in C([a, \sigma^2(b)], \mathbb{R})$ *satisfies*

$$\begin{cases} (\Phi(y^{\Delta}(t)))^{\Delta} \le 0, & t \in [a, b] \\ y(a) \ge 0, & y(\sigma^2(b)) \ge 0, \end{cases}$$

*then* $y(t) \ge 0$ *on* $[a, \sigma^2(b)]$.

*Proof.* We assume that $y(t) < 0$ for some $t \in (a, \sigma^2(b))$ and obtain a contradiction. Since $y(a) \ge 0$ and $y(\sigma^2(b)) \ge 0$, $y(t)$ would have a negative minimum in $(a, \sigma^2(b))$. Choose $t_0 \in (a, \sigma^2(b))$ such that

$$y(t_0) = \min\{y(t) : a \le t \le \sigma^2(b)\} < 0$$

and

$$y(t) > y(t_0) \quad \text{for } t \in (t_0, \sigma^2(b)].$$

There are four cases to consider:

Case 1: $\rho(t_0) = t_0 = \sigma(t_0)$. Assume $y^{\Delta}(t_0) > 0$. Then $\lim_{t \to t_0} y^{\Delta}(t) = y^{\Delta}(t_0) > 0$. This implies that there exists $\delta > 0$ such that $y^{\Delta}(t) > 0$ on $(t_0 - \delta, t_0]$, i.e., $y$ is increasing on $(t_0 - \delta, t_0]$. But this contradicts the way $t_0$ was chosen. Assume $y^{\Delta}(t_0) \le 0$. Then $\Phi(y^{\Delta}(t_0)) \le 0$. Since $(\Phi(y^{\Delta}(t)))^{\Delta} \le 0$ on $[a, b]$ and integrating it from $t_0$ to $t$, $t \in [t_0, \sigma^2(b))$, we obtain

$$\Phi(y^{\Delta}(t)) \le \Phi(y^{\Delta}(t_0)) \le 0, \quad t \in [t_0, \sigma^2(b)).$$

This implies that $\Phi(y^{\Delta}(t)) \le 0$ on $[t_0, \sigma^2(b))$ and so $y^{\Delta}(t) \le 0$ on $[t_0, \sigma^2(b))$. But we obtain $y(t_0) \ge y(\sigma^2(b)) \ge 0$, which contradicts the way $t_0$ is chosen.

Case 2: $\rho(t_0) = t_0 < \sigma(t_0)$. Assume $y^{\Delta}(t_0) > 0$. Then $\lim_{t \to t_0^-} y^{\Delta}(t) = y^{\Delta}(t_0) > 0$. This implies that there exists $\delta > 0$ such that $y^{\Delta}(t) > 0$ on $(t_0 - \delta, t_0]$, i.e., $y$ is increasing on $(t_0-\delta, t_0]$. But this contradicts the way $t_0$ was chosen. Assume $y^\Delta(t_0) \le 0$. Then this implies that $y^\sigma(t_0) \le y(t_0)$. But this contradicts with the way we picked $t_0$.

Case 3: $\rho(t_0) < t_0 < \sigma(t_0)$. Assume $y^\Delta(t_0) \le 0$. Then $y^\sigma(t_0) \le y(t_0)$. This contradicts the way $t_0$ was chosen. Assume $y^\Delta(t_0) > 0$. Since $\rho(t_0)$ is right-scattered, $y^\Delta(\rho(t_0)) < 0$ and so $\Phi(y^\Delta(\rho(t_0))) < 0$. However, since $(\Phi(y^\Delta(t)))^\Delta \le 0$ on $[a,b]$ and integrating it from $\rho(t_0)$ to $t$, $t \in [\rho(t_0), \sigma^2(b))$, we obtain

$$\Phi(y^\Delta(t)) \le \Phi(y^\Delta(\rho(t_0))) < 0, \quad t \in [\rho(t_0), \sigma^2(b)).$$

Therefore, $\Phi(y^\Delta(t)) < 0$ for $t \in [\rho(t_0), \sigma^2(b))$ and so $y^\Delta(t) < 0$ on $[\rho(t_0), \sigma^2(b))$. But this contradicts that $y^\Delta(t_0) > 0$.

Case 4: $\rho(t_0) < t_0 = \sigma(t_0)$. Assume $y^\Delta(t_0) \le 0$. Then $\Phi(y^\Delta(t_0)) \le 0$. Since $(\Phi(y^\Delta(t)))^\Delta \le 0$, $t \in [a,b]$ and integrating it from $t_0$ to $t$, $t \in [t_0, \sigma^2(b))$, we obtain

$$\Phi(y^\Delta(t)) \le \Phi(y^\Delta(t_0)) \le 0, \quad t \in [t_0, \sigma^2(b)).$$

This implies that $y^\Delta(t) \le 0$, $t \in [t_0, \sigma^2(b))$ and $y(t_0) \ge y(\sigma^2(b)) \ge 0$, which is a contradiction the way $t_0$ is chosen. Assume $y^\Delta(t_0) > 0$. Since $\rho(t_0)$ is right-scattered, $y^\Delta(\rho(t_0)) < 0$ and so $\Phi(y^\Delta(\rho(t_0))) < 0$. However, since $(\Phi(y^\Delta(t)))^\Delta \le 0$ on $[a,b]$ and integrating it from $\rho(t_0)$ to $t$, $t \in [\rho(t_0), \sigma^2(b))$, we obtain

$$\Phi(y^\Delta(t)) \le \Phi(y^\Delta(\rho(t_0))) < 0, \quad t \in [\rho(t_0), \sigma^2(b)).$$

Therefore, $\Phi(y^\Delta(t)) < 0$ for $t \in [\rho(t_0), \sigma^2(b))$ and so $y^\Delta(t) < 0$ on $[\rho(t_0), \sigma^2(b))$. But this contradicts that $y^\Delta(t_0) > 0$. □

The following lemma is known as "a weak comparison principle" in the literature.

**Lemma 3.4.** *If $u, v \in C([a, \sigma^2(b)], \mathbb{R})$ satisfy*

$$\begin{cases} (\Phi(u^\Delta(t)))^\Delta \le (\Phi(v^\Delta(t)))^\Delta, & t \in [a,b] \\ u(a) \ge v(a), \quad u(\sigma^2(b)) \ge v(\sigma^2(b)), \end{cases}$$

*then $u(t) \ge v(t)$ on $[a, \sigma^2(b)]$.*

*Proof.* Assume that $w(t) := u(t) - v(t) < 0$ for some $t \in [a, \sigma^2(b)]$. Since $w(a) \ge 0$ and $w(\sigma^2(b)) \ge 0$, $w(t)$ would have a negative minimum at a point $t_0 \in (a, \sigma^2(b))$. Choose $t_0 \in (a, \sigma^2(b))$ such that

$$w(t_0) = \min\{w(t) : a \le t \le \sigma^2(b)\} < 0$$

and

$$w(t) > w(t_0) \quad \text{for } t \in (t_0, \sigma^2(b)].$$

There are four cases to consider depending on what kind of point $t_0$ is. We only show how we get a contradiction for dense points and others can be shown as in the proof of Lemma 3.3. Let $\rho(t_0) = t_0 = \sigma(t_0)$. Assume $w^\Delta(t_0) > 0$. Then $\lim_{t\to t_0} w^\Delta(t) = w^\Delta(t_0) > 0$. This implies that there exists $\delta > 0$ such that $w^\Delta(t) > 0$ on $(t_0 - \delta, t_0]$, i.e., $w$ is increasing on $(t_0 - \delta, t_0]$. But this contradicts the way $t_0$ was chosen. Assume $w^\Delta(t_0) < 0$. Then $\lim_{t\to t_0} w^\Delta(t) = w^\Delta(t_0) < 0$. This implies that there exists $\delta > 0$ such that $w^\Delta(t) < 0$ on $[t_0, t_0 + \delta)$, i.e., $w$ is decreasing on $[t_0, t_0 + \delta)$. But this contradicts the way $t_0$ was chosen. Assume $w^\Delta(t_0) = 0$. Then $u^\Delta(t_0) = v^\Delta(t_0)$ and so $\Phi(u^\Delta(t_0)) = \Phi(v^\Delta(t_0))$. Since $(\Phi(u^\Delta(t)))^\Delta \le (\Phi(v^\Delta(t)))^\Delta$ on $[a, b]$ and integrating it from $t_0$ to $t$, $t \in [t_0, \sigma^2(b))$, we obtain

$$\Phi(u^\Delta(t)) - \Phi(u^\Delta(t_0)) \le \Phi(v^\Delta(t)) - \Phi(v^\Delta(t_0)), \quad t \in [t_0, \sigma^2(b)),$$

or

$$\Phi(u^\Delta(t)) - \Phi(v^\Delta(t)) \le \Phi(u^\Delta(t_0)) - \Phi(v^\Delta(t_0)) = 0, \quad t \in [t_0, \sigma^2(b)).$$

This implies that $u^\Delta(t) \le v^\Delta(t)$ on $[t_0, \sigma^2(b))$ and so $w^\Delta(t) \le 0$ on $[t_0, \sigma^2(b))$. But then we obtain $w(t_0) \ge w(\sigma^2(b)) \ge 0$, which contradicts the way $t_0$ is chosen. □

Throughout the next section we are interested in the existence of solutions of BVP (2). The method of lower and upper solutions is used. The approach is based on the Brouver and the Schauder fixed point theorems.

**Definition 3.1.** A function $\alpha(t) : [a, \sigma^2(b)] \mapsto \mathbb{R}$ is said to be a *lower solution* of BVP (2) if

$$\begin{cases} (\Phi(\alpha^\Delta(t)))^\Delta + F(t, \alpha^\sigma(t)) \ge 0, & t \in [a, b] \\ \alpha(a) \le A, \quad \alpha(\sigma^2(b)) \le B. \end{cases}$$

The definition of an *upper solution* $\beta$ of BVP (2) is given by reversing the above inequalities.

**Theorem 3.1.** *Let $\alpha$ and $\beta$ be a lower and an upper solution of BVP (2) such that $\alpha(t) \le \beta(t)$ on $[a, \sigma^2(b)]$. Then BVP (2) has a solution $y(t)$ such that $\alpha(t) \le y(t) \le \beta(t)$ on $[a, \sigma^2(b)]$.*

*Proof.* We consider the modified BVP

$$\begin{cases} (\Phi(y^\Delta(t)))^\Delta + F^*(t, y^\sigma(t)) = 0, & t \in [a, b] \\ y(a) = A, \quad y(\sigma^2(b)) = B, \end{cases} \tag{9}$$

where

$$F^*(t,x) = \begin{cases} F(t, \alpha^\sigma(t)) - \dfrac{x - \alpha^\sigma(t)}{1 + |x|}, & \text{if } x \le \alpha^\sigma(t), \\ F(t, x), & \text{if } \alpha^\sigma(t) \le x \le \beta^\sigma(t), \\ F(t, \beta^\sigma(t)) - \dfrac{x - \beta^\sigma(t)}{1 + |x|}, & \text{if } x \ge \beta^\sigma(t). \end{cases}$$

Then $F^*(t,x) : [a,b] \times \mathbb{R} \mapsto \mathbb{R}$ is continuous in $x$. Moreover, there exits $H > 0$ such that

$$|F^*(t,x)| \leq H \quad \text{for all } (t,x) \in [a,b] \times \mathbb{R}. \tag{10}$$

Equip $E = \{y : y \in C([a,\sigma^2(b)], \mathbb{R})\}$ with norm $||y|| = \max_{t \in [a,\sigma^2(b)]} \{|y(t)|\}$. Then $E$ is a Banach Space. Define the operator $A : E \mapsto E$ by

$$(Ay)(t) = \begin{cases} A, & \text{if } t = a, \\ B + \displaystyle\int_t^{\sigma(b)} \Phi^{-1}\left(\tau + \int_a^s F^*(r, y^\sigma(r))\Delta r\right)\Delta s, & \text{if } t \in (a, \sigma^2(b)), \\ B, & \text{if } t = \sigma^2(b), \end{cases}$$

where $\tau$ is a solution of the equation

$$\omega(\tau) := \Phi^{-1}(\tau) + \int_a^{\sigma(b)} \Phi^{-1}(\tau + \int_a^s F^*(r, y^\sigma(r))\Delta r)\Delta s = A - B. \tag{11}$$

It can be shown as in [9] that $A$ is well-defined, bounded and continuous. Therefore, $A$ has at least one fixed point in $E$ by the Brouver fixed point theorem. Let $y$ be a fixed point of $A$. Then

$$y^\Delta(t) = \begin{cases} -\Phi^{-1}(\tau), & \text{if } t = a, \\ -\Phi^{-1}\left(\tau + \displaystyle\int_a^t F^*(r, y^\sigma(r))\Delta r\right), & \text{if } t \in [a, \sigma^2(b)), \end{cases} \tag{12}$$

and one can show that $y$ is a solution of BVP (9). To complete the proof of theorem we only need to show that the solution $y(t)$ in (12) of (9) satisfies $\alpha(t) \leq y(t) \leq \beta(t)$ on $[a, \sigma^2(b)]$. To show that $y(t) \leq \beta(t)$ on $[a, \sigma^2(b)]$, we let $x(t) := y(t) - \beta(t)$ and suppose that $y(t) > \beta(t)$ for some $t \in (a, \sigma^2(b))$ to obtain a contradiction. Since $x(a) = y(a) - \beta(a) \leq A - A = 0$ and $x(\sigma^2(b)) = y(\sigma^2(b)) - \beta(\sigma^2(b)) \leq B - B = 0$, there exists $t_0 \in (a, \sigma^2(b))$ such that

$$x(t_0) = \max_{t \in [a,\sigma^2(b)]} x(t) > 0$$

and

$$x(t) < x(t_0) \text{ for } t \in (t_0, \sigma^2(b)].$$

It can be shown as in [3] that $t_0$ cannot be a left-dense and right-scattered point and so $\sigma(\rho(t_0)) = t_0$ and for other cases $(\Phi(y^\Delta(\rho(t_0))))^\Delta \leq (\Phi(\beta^\Delta(\rho(t_0))))^\Delta$.

On the other hand,

$$\begin{aligned}(\Phi(y^{\Delta}(\rho(t_0))))^{\Delta} &= -F^*(\rho(t_0), y^{\sigma}(\rho(t_0)))\\ &= -\left[F(\rho(t_0), \beta^{\sigma}(\rho(t_0))) - \frac{y^{\sigma}(\rho(t_0)) - \beta^{\sigma}(\rho(t_0))}{1+|y^{\sigma}(\rho(t_0))|}\right]\\ &= -\left[F(\rho(t_0), \beta^{\sigma}(\rho(t_0))) - \frac{y(t_0) - \beta(t_0)}{1+|y(t_0)|}\right]\\ &> -F(\rho(t_0), \beta^{\sigma}(\rho(t_0)))\\ &\geq (\Phi(\beta^{\Delta}(\rho(t_0))))^{\Delta}.\end{aligned}$$

But this gives us a contradiction. Therefore, $y(t) \leq \beta(t)$ on $[a, \sigma^2(b)]$. Similarly, we can show that $\alpha(t) \leq y(t)$ on $[a, \sigma^2(b)]$ and this completes the proof. □

**Lemma 3.5.** *Assume that $F(t, y) : [a, b] \times \mathbb{R} \mapsto \mathbb{R}$ is continuous and there exists $h \in C([a, b], [0, \infty))$ such that*

$$|F(t, y)| \leq h(t) \text{ for } t \in [a, b].$$

*Then BVP (2) has a solution $y \in C([a, \sigma^2(b)], \mathbb{R})$.*

*Proof.* Solving BVP (2) is same as the fixed point problem $Ay = y$. Since $A : C([a, \sigma^2(b)], \mathbb{R}) \mapsto C([a, \sigma^2(b)], \mathbb{R})$ is continuous and compact, the result follows from Schauder's fixed point theorem. □

## 4. Existence Theory

In this section, we assume that $\mathbb{T}$ has isolated points. We obtain existence of solutions of BVP (1), where nonlinear $f$ may change sign.

**Theorem 4.1.** *Let $n_0 \in \mathbb{Z}^+$ be fixed and assume the following conditions:*

(i) *$f(t, x) : [a, b] \times (0, \infty) \mapsto \mathbb{R}$ is continuous;*
(ii) *$q \in C([a, b], (0, \infty))$;*
(iii) *there exists a function $\alpha \in C([a, \sigma^2(b)], \mathbb{R})$ with $\alpha(a) = \alpha(\sigma^2(b)) = 0$, $\alpha > 0$ on $(a, \sigma^2(b))$ such that*

$$q(t)f(t, \alpha^{\sigma}(t)) \geq -(\Phi(\alpha^{\Delta}(t)))^{\Delta}, \quad t \in [a, b];$$

(iv) *there exists a function $\beta \in C([a, \sigma^2(b)], \mathbb{R})$ with $\beta(t) \geq \alpha(t)$ and $\beta(t) \geq \frac{1}{n_0}$, $t \in [a, \sigma^2(b)]$ such that*

$$q(t)f(t, \beta^{\sigma}(t)) \leq -(\Phi(\beta^{\Delta}(t)))^{\Delta}, \quad t \in [a, b].$$

*Then BVP (1) has a solution $y \in C([a, \sigma^2(b)], \mathbb{R})$ such that $\alpha(t) \leq y(t) \leq \beta(t)$ on $[a, \sigma^2(b)]$.*

*Proof.* Consider the BVP

$$(13)\qquad \begin{cases} -(\Phi(y^{\Delta}(t)))^{\Delta} = q(t)f^*_{n_0}(t, y^{\sigma}(t)), & t \in [a,b] \\ y(a) = y(\sigma^2(b)) = \dfrac{1}{n_0}, \end{cases}$$

where

$$f^*_{n_0}(t,x) = \begin{cases} f(t,\alpha^{\sigma}(t)) - \dfrac{x-\alpha^{\sigma}(t)}{1+|x|}, & \text{if } x \le \alpha^{\sigma}(t), \\ f(t,x), & \text{if } \alpha^{\sigma}(t) \le x \le \beta^{\sigma}(t), \\ f(t,\beta^{\sigma}(t)) - \dfrac{x-\beta^{\sigma}(t)}{1+|x|}, & \text{if } x \ge \beta^{\sigma}(t). \end{cases}$$

By Lemma (3.5), BVP (13) has a solution $y_{n_0} \in C([a,\sigma^2(b)],\mathbb{R})$. We now show that $y_{n_0}(t) \ge \alpha(t)$ on $[a,\sigma^2(b)]$. We suppose that it is not true to get a contradiction. We know that $y_{n_0}(a) = \frac{1}{n_0} > \alpha(a) = 0$ and $y_{n_0}(\sigma^2(b)) = \frac{1}{n_0} > \alpha(\sigma^2(b)) = 0$. Therefore, there exists an interval $[c,d] \subset [a,\sigma^2(b)]$ such that $y_{n_0}(t) < \alpha(t)$ on $[\sigma(c),\sigma(d)]$ and $y_{n_0}(c) \ge \alpha(c)$, $y_{n_0}(\sigma^2(d)) \ge \alpha(\sigma^2(d))$. For $[c,d]$, we have

$$\begin{aligned} -(\Phi(y^{\Delta}_{n_0}(t)))^{\Delta} &= q(t)f^*_{n_0}(t,y^{\sigma}_{n_0}(t)) \\ &= q(t)\left(f(t,\alpha^{\sigma}(t) - \frac{y^{\sigma}_{n_0}(t)-\alpha^{\sigma}(t)}{1+|y_{n_0}|}\right) \\ &> q(t)f(t,\alpha^{\sigma}(t)) \\ &\ge -(\Phi(\alpha^{\Delta}(t)))^{\Delta}. \end{aligned}$$

By Lemma (3.4), $y_{n_0}(t) \ge \alpha(t)$ on $[c,\sigma^2(d)]$, which is a contradiction. One can similarly show that $y_{n_0}(t) \le \beta(t)$ on $[a,\sigma^2(b)]$. Hence,

$$\alpha(t) \le y_{n_0}(t) \le \beta(t), \quad t \in [a,\sigma^2(b)].$$

Now proceed inductively to construct $y_{n_0+1}$, $y_{n_0+2}$, $\cdots$ as follows: We suppose for some $k \in \{n_0, n_0+1, \cdots\}$ with

$$\alpha(t) \le y_k(t) \le y_{k-1}(t), \quad t \in [a,\sigma^2(b)].$$

Here $y_{n_0-1} = \beta$. Then we consider the BVP

$$(14)\qquad \begin{cases} -(\Phi(y^{\Delta}(t)))^{\Delta} = q(t)f^*_{k+1}(t, y^{\sigma}(t)), & t \in [a,b] \\ y(a) = y(\sigma^2(b)) = \dfrac{1}{k+1}, \end{cases}$$

where

$$f^*_{k+1}(t,x) = \begin{cases} f(t,\alpha^{\sigma}(t)) - \dfrac{x-\alpha^{\sigma}(t)}{1+|x|}, & \text{if } x \le \alpha^{\sigma}(t), \\ f(t,x), & \text{if } \alpha^{\sigma}(t) \le x \le y^{\sigma}_k(t), \\ f(t,y^{\sigma}_k(t)) - \dfrac{x-y^{\sigma}_k(t)}{1+|x|}, & \text{if } x \ge y^{\sigma}_k(t). \end{cases}$$

By Lemma (3.5), BVP (14) has a solution $y_{k+1} \in C([a, \sigma^2(b)], \mathbb{R})$ and

$$\alpha(t) \le y_{k+1}(t) \le y_k(t) \quad \text{for } t \in [a, \sigma^2(b)].$$

Therefore, we have

$$\alpha(t) \le y_n(t) \le y_{n-1}(t) \le \cdots \le y_{n_0}(t) \le \beta(t) \quad \text{for } t \in [a, \sigma^2(b)].$$

Bolzano's theorem guarantees that $y_n \to y$ as $n \to \infty$. Also $y(a) = y(\sigma^2(b)) = 0$. Now $y_n(t) \ge \alpha(t) > 0$ on $t \in [a, \sigma^2(b)]$. Fix $t \in [a, \sigma^2(b)]$ and obtain

$$\begin{aligned} (\Phi(y_n^\Delta(t)))^\Delta &= \frac{\Phi^\sigma(y_n^\Delta(t)) - \Phi(y_n^\Delta(t))}{\mu(t)} \\ &\to \frac{\Phi^\sigma(y^\Delta(t)) - \Phi(y^\Delta(t))}{\mu(t)} \text{ as } n \to \infty \\ &= (\Phi(y^\Delta(t)))^\Delta, \end{aligned}$$

and $f(t, y_n(t)) \to f(t, y(t))$, $t \in [a, b]$ as $n \to \infty$. Therefore, $(\Phi(y^\Delta(t)))^\Delta + q(t)f(t, y^\sigma(t)) = 0$, $t \in [a, b]$ and $y(a) = y(\sigma^2(b)) = 0$. As a result, $y \in C([a, \sigma^2(b)], \mathbb{R})$ is a solution of BVP (1) and also we have

$$\alpha(t) \le y(t) \le \beta(t), \quad t \in [a, \sigma^2(b)].$$

□

**Theorem 4.2.** *In addition to (i-iii) of Theorem (4.1) assume*

(i) $q(t)f(t, y^\sigma(t)) \ge -(\Phi(\alpha^\Delta(t)))^\Delta$ *for* $(t, y) \in [a, b] \times \{y \in (0, \infty) : y < \alpha(t)\}$;

(ii) *there exists a function* $\beta \in C([a, \sigma^2(b)], \mathbb{R})$ *with* $\beta(t) \ge \frac{1}{n_0}$ *for* $t \in [a, \sigma^2(b)]$ *and*

$$q(t)f(t, \beta^\sigma(t)) \le -(\Phi(\beta^\Delta(t)))^\Delta, \quad t \in [a, b].$$

*Then BVP* (1) *has a solution* $y \in C([a, \sigma^2(b)], \mathbb{R})$ *such that* $\alpha(t) \le y(t) \le \beta(t)$, $t \in [a, \sigma^2(b)]$.

*Proof.* We now show that $\alpha(t) \le \beta(t)$ on $[a, \sigma^2(b)]$. Assume not, then since $\alpha(a) = 0 < \frac{1}{n_0} = \beta(a)$ and $\alpha(\sigma^2(b)) = 0 < \frac{1}{n_0} = \beta(\sigma^2(b))$, there exits $[c, d] \subset [a, \sigma^2(b)]$ such that $\beta(t) < \alpha(t)$ on $[\sigma(c), \sigma(d)]$ and $\beta(c) \ge \alpha(c)$, $\beta(\sigma^2(d)) \ge \alpha(\sigma^2(d))$. Therefore, we have

$$-(\Phi(\beta^\Delta(t)))^\Delta \ge q(t)f(t, \beta^\sigma(t)) \ge -(\Phi(\alpha^\Delta(t)))^\Delta,$$

by assumptions (i)-(ii). This implies that $(\Phi(\beta^\Delta(t)))^\Delta \le (\Phi(\alpha^\Delta(t)))^\Delta$ on $[c, d]$ and $\beta(c) \ge \alpha(c)$, $\beta(\sigma^2(d)) \ge \alpha(\sigma^2(d))$. By Lemma (3.4), $\alpha(t) \le \beta(t)$ on $[c, \sigma^2(d)]$, which is a contradiction. Hence, $\alpha(t) \le \beta(t)$ on $[a, \sigma^2(b)]$. By Theorem 4.1, the proof is completed. □

Now we construct the lower solution $\alpha$ in (iii) of Theorem 4.1, and in (i) of Theorem 4.2.

**Theorem 4.3.** *Let $n_0 \in \mathbb{Z}^+$ be fixed. In addition to Theorem 4.1 (i)-(ii) and Theorem 4.2 (ii), assume the following condition:*

$$\begin{cases} \text{let } n \in [n_0, \infty) \cup \mathbb{Z}^+ \text{ and associated with each } n \text{ there exists a constant } k_0 > 0 \\ \quad \text{such that for } t \in [a,b] \text{ and } 0 < y \le \frac{1}{n} \text{ we have } q(t) f(t, y^\sigma(t)) \ge k_0. \end{cases}$$

*Then BVP (1) has a solution $y \in C([a, \sigma^2(b)], \mathbb{R})$ such that $y(t) > 0$ on $(a, \sigma^2(b))$.*

*Proof.* Let $\alpha(t) = kv(t)$, $t \in [a, \sigma^2(b)]$, where $v \in C([a, \sigma^2(b)], [0, \infty))$ is a solution of

$$\begin{cases} (\Phi(v^\Delta(t)))^\Delta + 1 = 0, \quad t \in [a,b] \\ \qquad v(a) = 0 = v(\sigma^2(b)), \end{cases}$$

where $0 < k < \min\{k_0^{1/(p-1)}, \frac{1}{n_0 \|v\|}\}$. Since $(\Phi(v^\Delta(t)))^\Delta < 0$ on $[a,b]$, $v^{\Delta\Delta}(t) < 0$ on $[a, b]$. By Lemma 3.1, $v(t) \ge r(t)\|v\|$, $t \in [a, \sigma^2(b)]$, where $r(t)$ is defined as in (6). Since $\alpha(t) = kv(t)$, $\alpha(t) \le \frac{1}{n_0}$. For $0 < y < \alpha$, we have

$$-(\Phi(\alpha^\Delta(t)))^\Delta = -k^{p-1}(\Phi(v^\Delta(t)))^\Delta = k^{p-1} \le k_0 \le q(t) f(t, y^\sigma(t))$$

by the last assumption. This implies that (i) of Theorem 4.2 for $t \in [a, b]$. Since $\alpha(a) = \alpha(\sigma^2(b)) = 0$ and $\alpha > 0$, $t \in (a, \sigma^2(b))$, (iii) of Theorem 4.1 holds. By Theorem 4.2, there exists a solution $y \in C([a, \sigma^2(b)], \mathbb{R})$ such that $y > 0$ for $t \in (a, \sigma^2(b))$. □

In the next result we replace (ii) of Theorem 4.2 with a growth condition.

**Theorem 4.4.** *Let $n_0 \in \mathbb{Z}^+$ be fixed and suppose (i-iii) of Theorem 4.4 hold. In addition we assume the following:*

(i) *$|f(t,y)| \le g(y) + h(y)$ on $[a, b] \times (0, \infty)$, where $g(t) > 0$ is continuous and nonincreasing on $(0, \infty)$, $h(t) \ge 0$ is continuous on $[0, \infty)$, and $\frac{h(t)}{g(t)}$ is non-decreasing on $(0, \infty)$;*

(ii) *there exists a constant $M > \sup\{\alpha^\sigma(t) : t \in (a, \sigma^2(b))\}$ such that*

$$b_0 < \frac{1}{\Phi^{-1}(1 + \frac{h(M)}{g(M)})} \int_0^M \frac{dy}{\Phi^{-1}(g(y))},$$

*where*

$$b_0 = \max_{t \in [a,b]} \left( \int_a^{\rho(t)} \Phi^{-1}\Big(\int_s^t q(r)\Delta r\Big)\Delta s, \int_t^{\sigma(b)} \Phi^{-1}\Big(\int_{\rho(t)}^s q(r)\Delta r\Big)\Delta s \right).$$

*Then BVP (1) has a solution $y \in C([a, \sigma^2(b)], \mathbb{R})$ such that $\alpha(t) \le y(t) \le \beta(t)$ for $t \in [a, \sigma^2(b)]$.*

*Proof.* Choose $\epsilon > 0$, $\epsilon < M$ such that

$$\frac{1}{\Phi^{-1}(1+\frac{h(M)}{g(M)})}\int_{\epsilon}^{M}\frac{dy}{\Phi^{-1}(g(y))} > b_0. \tag{15}$$

Without loss of generality, $\frac{1}{n_0} < \epsilon$. We will consider the BVP

$$\begin{cases} (\Phi(y^{\Delta}(t)))^{\Delta} + q(t)g(y^{\sigma}(t))\left(1+\dfrac{h(M)}{g(M)}\right) = 0, & t \in [a,b] \\ y(a) = y(\sigma^2(b)) = \dfrac{1}{n_0}, \end{cases} \tag{16}$$

but first we consider the modification BVP

$$\begin{cases} (\Phi(y^{\Delta}(t)))^{\Delta} + q(t)g^*(y^{\sigma}(t))\left(1+\dfrac{h(M)}{g(M)}\right) = 0, & t \in [a,b] \\ y(a) = y(\sigma^2(b)) = \dfrac{1}{n_0}, \end{cases} \tag{17}$$

where

$$g^*(y) = \begin{cases} g(\frac{1}{n_0}), & \text{if } y \le \frac{1}{n_0}, \\ g(y), & \text{if } y \ge \frac{1}{n_0}, \end{cases}$$

Now $|g^*(y)| = g^*(y) \le g(\frac{1}{n_0})$ for $y \in \mathbb{R}$. By Lemma 3.5, BVP (17) has a solution $\beta \in C([a,\sigma^2(b)],\mathbb{R})$. Let $u(t) = \beta(t) - \frac{1}{n_0}$ for $t \in [a,\sigma^2(b)]$. Then $(\Phi(u^{\Delta}(t)))^{\Delta} = (\Phi(\beta^{\Delta}(t)))^{\Delta} \le 0$ for $t \in [a,b]$, $u(a) = 0$ and $u(\sigma^2(b)) = 0$. By Lemma 3.3, $u(t) \ge 0$ for $[a,\sigma^2(b)]$ and $\beta(t) \ge \frac{1}{n_0}$ for $[a,\sigma^2(b)]$. Then $\beta$ is a solution of BVP (16) as well. Now we show that $\alpha(t) \le \beta(t) \le M$, $t \in [a,\sigma^2(b)]$. We first show that $\beta(t) \ge \alpha(t)$ for $t \in [a,\sigma^2(b)]$. Assume not, then there exists $[c,d] \subset [a,\sigma^2(b)]$ such that $\beta(t) < \alpha(t)$ on $[\sigma(c),\sigma(d)]$, $\beta(c) \ge \alpha(c)$ and $\beta(\sigma^2(d)) \ge \alpha(\sigma^2(d))$ since $\beta(a) = \beta(\sigma^2(b)) = \frac{1}{n_0} > \alpha(a) = \alpha(\sigma^2(b)) = 0$. For $t \in [c,d]$,

$$\begin{aligned} -(\Phi(\beta^{\Delta}(t)))^{\Delta} &= q(t)g(\beta^{\sigma}(t))(1+\frac{h(M)}{g(M)}) \\ &\ge q(t)g(\alpha^{\sigma}(t))(1+\frac{h(\alpha^{\sigma}(t))}{g(\alpha^{\sigma}(t))}) \\ &\ge q(t)f(t,\alpha^{\sigma}(t)) \\ &\ge -(\Phi(\alpha^{\Delta}(t)))^{\Delta}. \end{aligned}$$

Since $\beta(c) \ge \alpha(c)$ and $\beta(\sigma^2(d)) \ge \alpha(\sigma^2(d))$, $\beta(t) \ge \alpha(t)$ on $[c,\sigma^2(d)]$ by Lemma 3.4 but this gives us a contradiction. Therefore, $\beta(t) \ge \alpha(t)$ on $[a,\sigma^2(b)]$. Second, we show that $\beta(t) \le M$ for $t \in [a,\sigma^2(b)]$. Since $(\Phi(\beta^{\Delta}(t)))^{\Delta} \le 0$ on $[a,b]$, $\beta^{\Delta\Delta}(t) \le 0$ on $[a,b]$ and so $\beta(t) \ge \frac{1}{n_0}$, $t \in [a,\sigma^2(b)]$. It implies that there exists $t_0 \in (a,\sigma^2(b))$ such that $\beta(t_0) = \max_{t\in[a,\sigma^2(b)]} \beta(t) > 0$, $\beta(t) < \beta(t_0)$ for $t \in (t_0, \sigma^2(b)]$. Hence, $\beta^\Delta(t) \leq 0$ on $[t_0, \sigma^2(b))$, $\beta^\Delta(t) \geq 0$ on $[a, t_0)$, and $\beta(t_0) = ||\beta|| > 0$. Also for $s \in [a, b]$, we have

$$-(\Phi(\beta^\Delta(s)))^\Delta = g(\beta^\sigma(s))(1 + \frac{h(M)}{g(M)})q(s). \tag{18}$$

Integrating equation (18) from $\tau$ to $t_0$ gives us

$$-\Phi(\beta^\Delta(t_0)) + \Phi(\beta^\Delta(\tau)) = (1 + \frac{h(M)}{g(M)}) \int_\tau^{t_0} g(\beta^\sigma(s))q(s)\Delta s.$$

Since $\beta^\Delta(t_0) \leq 0$ and from [4, Theorem 1.79], we have

$$\Phi(\beta^\Delta(\tau)) \leq (1 + \frac{h(M)}{g(M)}) \sum_{s\in[\tau,t_0)} \mu(s)g(\beta^\sigma(s))q(s)$$

Since $g(\beta^\sigma(\tau)) \leq g(u) \leq g(\beta(\tau))$ for $\beta(\tau) \leq u \leq \beta^\sigma(\tau)$ when $\tau < t_0$, This implies that

$$\Phi(\beta^\Delta(\tau)) \leq (1 + \frac{h(M)}{g(M)})g(\beta^\sigma(\tau)) \sum_{s\in[\tau,t_0)} \mu(s)q(s),$$

i.e.,

$$\Phi(\beta^\Delta(\tau)) \leq (1 + \frac{h(M)}{g(M)})g(\beta^\sigma(\tau)) \int_\tau^{t_0} q(s)\Delta s.$$

and so we have

$$\frac{\beta^\Delta(\tau))}{\Phi^{-1}(g(\beta^\sigma(\tau)))} \leq \Phi^{-1}(1 + \frac{h(M)}{g(M)})\Phi^{-1}(\int_\tau^{t_0} q(s)\Delta s), \quad \tau < t_0. \tag{19}$$

Again since $g(\beta^\sigma(\tau)) \leq g(u) \leq g(\beta(\tau))$ for $\beta(\tau) \leq u \leq \beta^\sigma(\tau)$ when $\tau < t_0$, we have

$$\int_{\beta(\tau)}^{\beta^\sigma(\tau)} \frac{du}{\Phi^{-1}(g(u))} \leq \frac{1}{\Phi^{-1}(g(\beta^\sigma(\tau)))} \int_{\beta(\tau)}^{\beta^\sigma(\tau)} du = \frac{1}{\Phi^{-1}(g(\beta^\sigma(\tau)))}\mu(\tau)\beta^\Delta(\tau)$$

by (3). And we get

$$\frac{1}{\mu(\tau)} \int_{\beta(\tau)}^{\beta^\sigma(\tau)} \frac{du}{\Phi^{-1}(g(u))} \leq \Phi^{-1}(1 + \frac{h(M)}{g(M)})\Phi^{-1}(\int_\tau^{t_0} q(s))\Delta s$$

from (19). Integrating the above from $a$ to $\rho(t_0)$ gives us

$$\sum_{\tau\in[a,\rho(t_0))} \mu(\tau)\frac{1}{\mu(\tau)} \int_{\beta(\tau)}^{\beta^\sigma(\tau)} \frac{du}{\Phi^{-1}(g(u))} \leq \Phi^{-1}(1 + \frac{h(M)}{g(M)}) \sum_{\tau\in[a,\rho(t_0))} \mu(\tau)\Phi^{-1}(\int_\tau^{t_0} q(s)),$$

i.e.,

$$\int_{\frac{1}{n_0}}^{\beta(t_0)} \frac{du}{\Phi^{-1}(g(u))} \leq \Phi^{-1}(1 + \frac{h(M)}{g(M)}) \int_a^{\rho(t_0)} \Phi^{-1}(\int_\tau^{t_0} q(s)d\Delta s)\Delta\tau$$

by [4, Theorem 1.79]. Similarly, we integrate equation (18) from $\rho(t_0)$ to $\tau$ and obtain

$$-\Phi(\beta^\Delta(\tau)) = -\Phi(\beta^\Delta(\rho(t_0))) + (1 + \frac{h(M)}{g(M)}) \int_{\rho(t_0)}^{\tau} g(\beta^\sigma(s))q(s)\Delta s.$$

Since $\beta^{\Delta}(\rho(t_0)) \geq 0$ and $g(\beta^\sigma(s)) \leq g(\beta^\sigma(\tau))$ for $\beta^\sigma(s) \geq \beta^\sigma(\tau)$, we obtain

$$-\frac{\beta^{\Delta}(\tau)}{\Phi^{-1}(g(\beta^\sigma(\tau)))} \leq \Phi^{-1}(1+\frac{h(M)}{g(M)})\Phi^{-1}(\int_{\rho(t_0)}^{\tau} q(s)\Delta s). \tag{20}$$

Since $g(\beta^\sigma(\tau)) \leq g(u)$ for $u \leq \beta^\sigma(\tau) \leq \beta(\tau)$,

$$\int_{\beta^\sigma(\tau)}^{\beta(\tau)} \frac{du}{\Phi^{-1}(g(u))} \leq \frac{1}{\Phi^{-1}(g(\beta^\sigma(\tau)))}\int_{\beta^\sigma(\tau)}^{\beta(\tau)} du = -\frac{1}{\Phi^{-1}(g(\beta^\sigma(\tau)))}\mu(\tau)\beta^{\Delta}(\tau).$$

Hence

$$\frac{1}{\mu(\tau)}\int_{\beta^\sigma(\tau)}^{\beta(\tau)} \frac{du}{\Phi^{-1}(g(u))} \leq \frac{-\beta^{\Delta}(\tau)}{\Phi^{-1}(g(\beta^\sigma(\tau))} \leq \Phi^{-1}(1+\frac{h(M)}{g(M)})\Phi^{-1}(\int_{\rho(t_0)}^{\tau} q(s)\Delta s)$$

by (20). Integrating the above from $t_0$ to $\sigma(b)$ gives us

$$\sum_{\tau\in[t_0,\sigma(b))} \mu(\tau)\frac{1}{\mu(\tau)}\int_{\beta^\sigma(\tau)}^{\beta(\tau)} \frac{du}{\Phi^{-1}(g(u))} \leq \Phi^{-1}(1+\frac{h(M)}{g(M)})\sum_{\tau\in[t_0,\sigma(b))} \mu(\tau)\Phi^{-1}(\int_{\rho(t_0)}^{\tau} q(s)\Delta s),$$

i.e.,

$$\int_{\frac{1}{n_0}}^{\beta(t_0)} \frac{du}{\Phi^{-1}(g(u))} \leq \Phi^{-1}(1+\frac{h(M)}{g(M)})\int_{t_0}^{\sigma(b)} \Phi^{-1}(\int_{\rho(t_0)}^{\tau} q(s)\Delta s)\Delta\tau.$$

Altogether, we have

$$\int_{\epsilon}^{\beta(t_0)} \frac{du}{\Phi^{-1}(g(u))} \leq \int_{\frac{1}{n_0}}^{\beta(t_0)} \frac{du}{\Phi^{-1}(g(u))} \leq b_0\Phi^{-1}(1+\frac{h(M)}{g(M)}).$$

Hence, $||\beta|| = \beta(t_0) \leq M$ by (15). Notice that

$$f(t,\beta^\sigma(t)) \leq g(\beta^\sigma(t))(1+\frac{h(\beta^\sigma(t))}{g(\beta^\sigma(t))}) \leq g(\beta^\sigma(t))(1+\frac{h(M)}{g(M)}),$$

$t \in [a,b]$. Therefore, $\beta(t) \geq \dfrac{1}{n_0}$ and $\beta(t) \geq \alpha(t)$ on $[a,\sigma^2(b)]$ such that

$$-(\Phi(\beta^{\Delta}(t)))^{\Delta} = q(t)g(\beta^\sigma(t))(1+\frac{h(M)}{g(M)}) \geq q(t)f(t,\beta^\sigma(t)), \quad t\in[a,b].$$

So $\beta(t)$ satisfies (iv) of Theorem 4.1. By Theorem 4.1, BVP (1) has a solution $y \in \mathrm{C}([a,\sigma^2(b)],\mathbb{R})$ such that $\alpha(t) \leq y(t) \leq \beta(t)$ on $[a,\sigma^2(b)]$. □

We finish this section with following result.

**Theorem 4.5.** *Let $n_0 \in \mathbb{Z}^+$ be fixed and suppose (i-ii) of Theorem 4.1, the last assumption of Theorem 4.3 and (i) of Theorem 4.4 hold. In addition assume there exists a constant $M > 0$ such that (ii) of Theorem 4.4 holds. Then BVP (1) has a solution $y \in \mathrm{C}([a,\sigma^2(b)],\mathbb{R})$ such that $y(t) > 0$ on $(a,\sigma^2(b))$.*

*Proof.* If we show that there exists $\alpha \in C([a, \sigma^2(b)], \mathbb{R})$ such that (iii) of Theorem 4.1 holds and $M > \alpha(t)$ on $[a, \sigma^2(b)]$, then we are done by Theorem 4.4. Let $\alpha(t) = kv(t)$, $t \in [a, \sigma^2(b)]$, where $v \in C([a, \sigma^2(b)], [0, \infty))$ is a solution of

$$\begin{cases} (\Phi(v^\Delta(t)))^\Delta + 1 = 0, & t \in [a, b] \\ v(a) = 0 = v(\sigma^2(b)), \end{cases}$$

where $0 < k < \min\{k_0^{1/(p-1)}, \frac{1}{n_0||v||}, \frac{M}{||v||}\}$. Therefore, $\alpha(t) \leq \frac{1}{n_0}$, $-(\Phi(\alpha^\Delta(t)))^\Delta = k^{p-1} \leq k_0$, $\alpha(a) = \alpha(\sigma^2(b)) = 0$, $\alpha > 0$ for $t \in [\sigma(a), \sigma(b)]$ and (iii) of Theorem 4.1 holds since

$$q(t)f(t, \alpha^\sigma(t)) \geq k_0 \geq -(\Phi(\alpha^\Delta(t)))^\Delta, \quad t \in [a, b].$$

So $\alpha \in C([a, \sigma^2(b)], \mathbb{R})$, $\alpha(t) < M$ on $[a, \sigma^2(b)]$ and this completes the proof. □

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