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OSCILLATION CRITERIA FOR FOURTH ORDER NONLINEAR DIFFERENCE EQUATIONS

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Abstract. Some new criteria for the oscillation of the fourth order difference equation

$$\Delta^2 \left(a(n)(\Delta^2 x(n))^\alpha\right) + q(n)f(x(n+1)) = 0,$$

where α is the ratio of two positive odd integers are established.

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1. Introduction

This paper is concerned with the oscillatory behavior of the fourth order difference equation

$$\Delta^2 \left(a(n)(\Delta^2 x(n))^\alpha\right) + q(n)f(x(n+1)) = 0 \tag{1.1}$$

where $n \in \mathbb{N}_0 = \{n_0, n_0+1, \dots\}$, n_0 is a nonnegative integer, Δ is the forward difference operator defined by $\Delta x(n) = x(n+1) - x(n)$, and α is the ratio of two positive odd integers, $\{a(n)\}$, $\{q(n)\}$ are positive sequences, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $xf(x) > 0$ and $f'(x) \geq 0$ for $x \neq 0$.

In what follows we shall assume that

$$\sum^{\infty} a^{-1/\alpha}(n) = \infty. \tag{1.2}$$

We introduce the operators L_i, $i = 0, 1, 2, 3, 4$, as follows:

$$L_0 x = x,\ L_1 x = \Delta L_0 x,\ L_2 x = a(\Delta L_1 x)^\alpha,\ L_3 x = \Delta L_2 x,\ L_4 x = \Delta L_3 x. \tag{1.3}$$

By a solution of equation (1.1), we mean a real sequence $\{x(n)\}$ satisfying equation (1.1) for all $n \geq n_0 - \tau + 1$. A nontrivial solution $\{x(n)\}$ of (1.1) is said to be nonoscillatory if it is either eventually positive or eventually negative, and it is oscillatory otherwise. The equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

In the last two decades there has been an increasing interest in studying oscillatory, nonoscillatory and asymptotic behavior of solutions of difference equations. Most of the work on the subject, however, has been restricted to first and second order linear, half-linear and nonlinear difference equations, as well as equations of type (1.1) with $\alpha = 1$. For recent contributions, we refer to [1–7, 9, 11, 12] and the references cited therein. However, it seems that little is known regarding the oscillation of equation (1.1). Therefore, the purpose of this paper is to establish a systematic study for the oscillation of equation

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(1.1). In Section 2, we shall give the proof of a critical lemma which is useful throughout this paper. Section 3 is devoted to the study of equation (1.1) when f satisfies different sets of conditions. In Section 4, we give the results for the oscillation of delay difference equation of type (1.1) with $f(x) = x^\beta$, where β is the ratio of two positive odd integers. We shall also establish some necessary and sufficient conditions for the bounded as well as unbounded oscillation of equations related to equation (1.1). Further, we shall investigate the oscillation of difference equations with advanced argument that are related to equation (1.1). In Section 5, we shall present a comparison criterion which allows us to extend the obtained results to more general equations of neutral type. We remark that the obtained results are presented in a form which is essentially new even for the special case when $\alpha = 1$.

2. Preliminaries

If $\{x(n)\}$ is an eventually positive solution of equation (1.1), then $L_4x(n) \leq 0$ eventually and since (1.2) holds, it follows that $L_ix(n)$, $i = 1, 2, 3$, are eventually of one sign. We need to distinguish the following two cases:

(I). $L_ix(n) > 0$, $i = 0, 1, 2, 3$, and $L_4x(n) \leq 0$ eventually.

(II). $L_0x(n) > 0$, $L_1x(n) > 0$, $L_2x(n) < 0$, $L_3x(n) > 0$, and $L_4x(n) \leq 0$ eventually.

Suppose that (I) holds. Since $L_3x(n) > 0$ is decreasing for $n \geq n_0$ (say), we obtain

$$L_2x(n) - L_2x(n_0) = \sum_{i=n_0}^{n-1} L_3x(i),$$

or

$$a(n)(\Delta L_1x(n))^\alpha \geq (n - n_0)L_3x(n-1) \geq (n - n_0)L_3x(n),$$

or

$$\Delta^2x(n) \geq \left(\frac{(n-n_0)}{a(n)}\right)^{1/\alpha} L_3^{1/\alpha}x(n) \quad \text{for} \quad n \geq n_0. \tag{2.1}$$

Summing (2.1) from n_0 to $n-1$, using (I) and the decreasing property of $L_3x(n)$, $n \geq n_0$, we have

$$\Delta x(n) \geq \left(\sum_{j=n_0}^{n-1} \left(\frac{(j-n_0)}{a(j)}\right)^{1/\alpha}\right) L_3^{1/\alpha}x(n), \quad n \geq n_0, \tag{2.2}$$

and

$$x(n) \geq \left(\sum_{s=n_0}^{n-1}\sum_{j=n_0}^{s-1} \left(\frac{(j-n_0)}{a(j)}\right)^{1/\alpha}\right) L_3^{1/\alpha}x(n), \quad n \geq n_0. \tag{2.3}$$

Suppose that (II) holds. Then, noting that $L_3x(n) > 0$ is decreasing and $L_2x(n) < 0$, from the equation

$$L_2x(2n) - L_2x(n) = \sum_{i=n}^{2n-1} L_3x(i),$$

we see that

$$-L_2x(n) \geq nL_3x(2n),$$

which is rewritten as

$$-\Delta^2 x(n) \geq \left(\frac{n}{a(n)}\right)^{1/\alpha} L_3^{1/\alpha} x(2n) \quad \text{for} \quad n \geq n_0. \tag{2.4}$$

Summing (2.4) from n to $2n-1$, we obtain

$$\Delta x(n) \geq \left(\sum_{j=n}^{2n-1} \left(\frac{j}{a(j)}\right)^{1/\alpha}\right) L_3^{1/\alpha} x(4n) \tag{2.5}$$

and

$$x(n) \geq \left(\sum_{s=n_0}^{n-1} \sum_{j=s}^{2s-1} \left(\frac{j}{a(j)}\right)^{1/\alpha}\right) L_3^{1/\alpha} x(4n). \tag{2.6}$$

For $n \geq n_0$, we let

$$h(n, n_0; a) = \min\left\{\sum_{j=n_0}^{n-1} \left(\frac{j-n_0}{a(j)}\right)^{1/\alpha}, \sum_{j=n}^{2n-1} \left(\frac{j}{a(j)}\right)^{1/\alpha}\right\}$$

and

$$H(n, n_0; a) = \sum_{j=n_0}^{n-1} h(j, n_0; a).$$

Combining the above inequalities, we are ready to state the following crucial lemma.

Lemma 2.1. *Let $\{x(n)\}$ be a positive solution of equation* (1.1) *for $n \geq n_0$. Then, for all sufficiently large $n \geq n_0$,*

$$\Delta x(n) \geq h(n, n_0; a) L_3^{1/\alpha} x(4n)$$

and

$$x(n) \geq H(n, n_0; a) L_3^{1/\alpha} x(n).$$

We shall also need the following lemma.

Lemma 2.2 ([8]). *If X and Y are nonnegative, then*

$$X^\lambda - \lambda X Y^{\lambda-1} + (\lambda - 1) Y^\lambda \geq 0, \quad \lambda > 1,$$

where equality holds if and only if $X = Y$.

3. Oscillation Criteria

In what follows we shall assume that

$$f(u) - f(v) = g(u, v)(u - v) \quad \text{for} \quad u, v \neq 0, \tag{3.1}$$

where g is a nonnegative function, and

$$f^{(1/\alpha)-1}(u) g(u, v) \geq k > 0 \quad \text{for} \quad u, v \neq 0 \quad \text{and} \quad k \text{ is a constant.} \tag{3.2}$$

Our first result is embodied in the following theorem.

Theorem 3.1. *Let conditions* (1.2), (3.1) *and* (3.2) *hold, and assume that there exists a positive sequence* $\{\rho(n)\}$ *such that for* $n \geq N_0 > n_0$,

$$\limsup_{m\to\infty} \sum_{n=N_0}^{m} \left[\rho(n)q(4n) - \left(\frac{1}{k^\alpha}\frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\frac{(\Delta\rho(n))^{1+\alpha}}{(\rho(n)h(n,n_0;a))^\alpha}\right)\right] = \infty, \quad (3.3)$$

where $h(n, n_0; a)$ *is as in Lemma* 2.1. *Then, equation* (1.1) *is oscillatory.*

Proof. Assume for the sake of contradiction that equation (1.1) has a nonoscillatory solution $\{x(n)\}$ and that $\{x(n)\}$ is eventually positive. There exists a positive integer $n_1 \geq n_0$ such that $x(t) > 0$ for $n \geq n_1$. From equation (1.1), we see that $L_4x(t) \leq 0$ for $n \geq n_1$ and hence $L_ix(n)$, $i = 1, 2, 3$, are eventually of one sign for $n \geq n_1$. It is easy to check that $L_3x(n) > 0$ is decreasing and $L_1x(n) > 0$ for $n \geq n_2$ for some $n_2 \geq n_1$. By Lemma 2.1, there exists an integer $n_3 \geq n_2$ such that

$$\Delta x(n) \geq h(n, n_3; a)L_3^{1/\alpha}x(4n) \quad \text{for} \quad n \geq n_3. \quad (3.4)$$

Define

$$w(n) = \rho(n)\frac{L_3x(4n)}{f(x(n))} \quad \text{for} \quad n \geq n_3.$$

Then, for $n \geq n_3$, we have

$$\begin{aligned}\Delta w(n) &= \Delta\rho(n)\frac{L_3x(4n+4)}{f(x(n+1))} + \rho(n)\frac{\Delta L_3x(4n)}{f(x(n+1))}\\ &\quad - \rho(n)L_3x(4n)\frac{f(x(n+1)) - f(x(n))}{f(x(n+1))f(x(n))}\\ &= -\rho(n)q(4n)\frac{f(x(4n))}{f(x(n+1))} + \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1)\\ &\quad - \rho(n)L_3x(4n)\frac{\Delta x(n)g(x(n+1), x(n))}{f(x(n+1))f(x(n))}. \quad (3.5)\end{aligned}$$

Using (3.1), (3.2) and (3.4) in (3.5), we get

$$\begin{aligned}\Delta w(n) &\leq -\rho(n)q(4n) + \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1)\\ &\quad - k\frac{\rho(n)}{\rho^{1+(1/\alpha)}(n+1)}h(n, n_3; a)w^{1+(1/\alpha)}(n+1) \quad \text{for} \quad n \geq n_3. \quad (3.6)\end{aligned}$$

Set

$$X = (k\rho(n)h(n, n_3; a))^{\alpha/(\alpha+1)}\frac{w(n+1)}{\rho(n+1)}, \quad \lambda = \frac{\alpha+1}{\alpha} > 1,$$

and

$$Y = \left(\frac{\alpha}{\alpha+1}\right)^\alpha \left(\frac{\Delta\rho(n)}{\rho(n+1)}\right)^\alpha \left[(k\rho(n)h(n, n_3; a))^{-\alpha/(\alpha+1)}\rho(n+1)\right]^\alpha$$

in Lemma 2.2, to conclude that for $n \geq n_4 > n_3$,

$$\frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) - k\frac{\rho(n)}{\rho^{1+(1/\alpha)}(n+1)}h(n,n_3;a)w^{1+(1/\alpha)}(n+1) \leq \frac{1}{k^\alpha}\frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\frac{(\Delta\rho(n))^{1+\alpha}}{(\rho(n)h(n,n_3;a))^\alpha},$$

and therefore

$$\Delta w(n) \leq -\rho(n)q(4n) + \frac{1}{k^\alpha}\frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\frac{(\Delta\rho(n))^{1+\alpha}}{(\rho(n)h(n,n_3;a))^\alpha}, \quad n \geq n_4. \tag{3.7}$$

Summing both sides of (3.7) from n_4 to $m \geq n_4$, we obtain

$$w(m+1) - w(n_4) \leq -\sum_{n=n_4}^{m}\Bigg[\rho(n)q(4n) - \left(\frac{1}{k^\alpha}\frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\frac{(\Delta\rho(n))^{1+\alpha}}{(\rho(n)h(n,n_0;a))^\alpha}\right)\Bigg] \to -\infty \quad \text{as} \quad m \to \infty,$$

which contradicts the fact that $w(m) > 0$ for $m \geq n_3$. □

Next, we present the following interesting criteria for the oscillation of equation (1.1) with $0 < \alpha \leq 1$.

Theorem 3.2. *Let $0 < \alpha \leq 1$ and conditions (1.2), (3.1) and (3.2) hold. If there exists a positive sequence $\{\rho(n)\}$ such that*

$$0 < Q^*(n) = \rho(n)Q(n); \quad Q(n) = \sum_{j=n}^{\infty} q(4j) \tag{3.8}$$

and

$$\limsup_{n\to\infty} \sum_{j=N_0>n_0}^{n}\left[\rho(j)q(4j) - \frac{1}{4k}\frac{(\Delta\rho(j))^2 Q^{1-(1/\alpha)}(j+1)}{\rho(j)h(j,n_0;a)}\right] = \infty, \tag{3.9}$$

where $h(n,n_0;a)$ is as in Lemma 2.1, then equation (1.1) is oscillatory.

Proof. Let $\{x(n)\}$ be an eventually positive solution of equation (1.1). Define

$$y(n) = \frac{L_3x(4n)}{f(x(n))} \quad \text{for} \quad n \geq n_0.$$

Then, for $n \geq n_0$, we get

$$\Delta y(n) \leq -q(4n).$$

Summing the above inequality from n to u and letting $u \to \infty$, we find

$$y(n) \geq \sum_{j=n}^{\infty} q(4j) = Q(n), \quad n \geq n_0.$$

Next, we define

$$w(n) = \rho(n)\frac{L_3x(4n)}{f(x(n))}, \quad n \geq n_0.$$

Then,

$$w(n) \geq Q^*(n), \quad n \geq n_0. \tag{3.10}$$

Proceeding as in the proof of Theorem 3.1, we obtain (3.6) for $n \geq n_3$. Now, for $n \geq n_4$ for some $n_4 > n_3$, we obtain, for $n \geq n_4$,

$$\begin{aligned}
\Delta w(n) &\leq -\rho(n)q(4n) + \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) \\
&\quad - k\frac{\rho(n)}{\rho^{1+(1/\alpha)}(n+1)}\,h(n,n_3;a)w^2(n+1)w^{(1/\alpha)-1}(n+1) \\
&\leq -\rho(n)q(4n) + \frac{\Delta\rho(n)}{\rho(n+1)}\,w(n+1) \\
&\quad - k\rho(n)\rho^{-1-(1/\alpha)}(n+1)h(n,n_3;a)Q^{*(1/\alpha)-1}(n+1)w^2(n+1) \\
&= -\rho(n)q(4n) + \frac{1}{4k}\,\frac{(\Delta\rho(n))^2Q^{1-(1/\alpha)}(n+1)}{\rho(n)h(n,n_3;a)} \\
&\quad - \Bigg[\sqrt{k\rho(n)\rho^{-2}(n+1)h(n,n_3;a)Q^{(1/\alpha)-1}(n+1)}w(n+1) \\
&\quad - \frac{\Delta\rho(n)}{2\rho(n+1)\sqrt{k\rho(n)\rho^{-2}(n+1)h(n,n_3;a)Q^{(1/\alpha)-1}(n+1)}}\Bigg]^2 \\
&\leq -\rho(n)q(4n) + \frac{1}{4k}\,\frac{(\Delta\rho(n))^2Q^{1-(1/\alpha)}(n+1)}{\rho(n)h(n,n_3;a)}.
\end{aligned}$$

The rest of the proof is similar to that of Theorem 3.1 and hence omitted. □

The following result is concerned with the oscillation of a special case of equation (1.1), namely, the equation

$$\Delta^2\left(a(n)(\Delta^2x(n))^\alpha\right) + q(n)x^\alpha(n+1) = 0, \quad \alpha \geq 1. \tag{3.11}$$

Theorem 3.3. *Let $\alpha \geq 1$ and condition (1.2) hold. If there exists a positive sequence $\{\rho(n)\}$ such that*

$$\limsup_{n\to\infty} \sum_{j=N_0>n_0}^{n} \left[\rho(j)q(4j) - \frac{(\Delta\rho(n))^2}{4\alpha\rho(n)h^\alpha(n,n_0;a)(n-n_0)^{\alpha-1}}\right] = \infty, \tag{3.12}$$

where $h(n,n_0;a)$ is an in Lemma 2.1, then all bounded solutions of equation (3.11) are oscillatory.

Proof. Let $\{x(n)\}$ be an eventually bounded positive solution of equation (3.11). It is easy to see that $\{x(n)\}$ satisfies (II). Proceeding as in the proof of Theorem 3.1, we obtain (3.4), $n \geq n_3$, and we can easily see that

$$x(n) \geq (n-n_3)\Delta x(n), \quad n \geq n_3. \tag{3.13}$$

Define

$$w(n) = \rho(n)\,\frac{L_3x(4n)}{x^\alpha(n)}, \quad n \geq 3.$$

Then, for $n \geq n_3$, we have

$$\begin{aligned}
\Delta w(n) &\leq -\rho(n)q(4n) + \frac{\Delta\rho(n)}{\rho(n+1)} w(n+1) - \rho(n)L_3x(4n)\frac{x^\alpha(n+1) - x^\alpha(n)}{x^\alpha(n+1)x^\alpha(n)} \\
&\leq -\rho(n)q(4n) + \frac{\Delta\rho(n)}{\rho(n+1)} w(n+1) \\
&\quad - \alpha\rho(n)L_3x(4n)(\Delta x(n))^\alpha(\Delta x(n))^{1-\alpha}\frac{x^{\alpha-1}(n)}{(x^\alpha(n+1))^2} \\
&\leq -\rho(n)q(4n) + \frac{\Delta\rho(n)}{\rho(n+1)} w(n+1) \\
&\quad - \alpha\frac{\rho(n)}{\rho^2(n+1)} h^\alpha(n, n_3; a)\left(\frac{x(n)}{\Delta x(n)}\right)^{\alpha-1} w^2(n+1).
\end{aligned}$$

Using (3.13) in the above inequality, we get for $n \geq n_4 > n_3$,

$$\begin{aligned}
\Delta w(n) &\leq -\rho(n)q(4n) + \frac{\Delta\rho(n)}{\rho(n+1)} w(n+1) \\
&\quad - \alpha\frac{\rho(n)}{\rho^2(n+1)} h^\alpha(n, n_3; a)(n-n_3)^{\alpha-1} w^2(n+1) \\
&= -\rho(n)q(4n) + \frac{1}{4\alpha}\,\frac{(\Delta\rho(n))^2}{\rho(n)h^\alpha(n, n_3; a)(n-n_3)^{\alpha-1}} \\
&\quad - \left[\sqrt{\alpha\frac{\rho(n)}{\rho^2(n+1)}h^\alpha(n, n_3; a)(n-n_3)^{\alpha-1}}\,w(n+1)\right. \\
&\quad \left. - \frac{\Delta\rho(n)}{2\rho(n+1)\sqrt{\alpha\frac{\rho(n)}{\rho^2(n+1)}\,h^\alpha(n, n_3; a)(n-n_3)^{\alpha-1}}}\right]^2 \\
&\leq -\left[\rho(n)q(4n) - \frac{1}{4\alpha}\,\frac{(\Delta\rho(n))^2}{\rho(n)\,h^\alpha(n, n_3; a)(n-n_3)^{\alpha-1}}\right].
\end{aligned}$$

The rest of the proof is similar to that of Theorem 3.1, and hence omitted. □

Next, we present the following oscillation result which is for superlinear equations of type (1.1).

Theorem 3.4. *Let $\alpha > 1$ and condition (1.2) hold, and*

$$\int^{\infty} \frac{du}{f^{1/\alpha}(u)} < \infty \quad \text{and} \quad \int^{-\infty} \frac{du}{f^{1/\alpha}(u)} < \infty. \tag{3.14}$$

If there exists a positive sequence $\{\rho(n)\}$ such that

$$\Delta\rho(n) > 0 \quad \text{and} \quad \Delta(h^\alpha(n, n_0; a)\Delta\rho(n)) \leq 0 \quad \text{for} \quad n \geq n_0 \tag{3.15}$$

and

$$\sum^{\infty} \rho(n)q(4n) = \infty, \tag{3.16}$$

then equation (1.1) is oscillatory.

Proof. Assume that $\{x(n)\}$ is a nonoscillatory solution of equation (1.1), say, $x(n) > 0$ for $n \geq n_0 \geq 1$. As in the proof of Theorem 3.1, define the same $w(n)$ to obtain equation (3.5) and inequality (3.4) for $n \geq n_3$. Thus,

$$\begin{aligned}\Delta w(n) &\leq -\rho(n)q(4n) + \Delta\rho(n)\frac{L_3 x(4n+4)}{f(x(n+1))} \\ &\leq -\rho(n)q(4n) + \Delta\rho(n)\left(\frac{L_3^{1/\alpha}x(4n)}{f^{1/\alpha}(x(n+1))}\right)^{\alpha}, \quad n \geq n_3. \end{aligned} \tag{3.17}$$

Using (3.4) in (3.17), we obtain

$$\Delta w(n) \leq -\rho(n)q(4n) + \Delta\rho(n)h^{\alpha}(n, n_3; a)\left(\frac{\Delta x(n)}{f^{1/\alpha}(x(n+1))}\right)^{\alpha}. \tag{3.18}$$

Summing (3.18) from $n_4 > n_3$ to $m \geq n_4$, we get

$$\begin{aligned} &w(m+1) - w(n_4) \\ &\quad\leq -\sum_{j=n_4}^{m}\rho(j)q(4j) + h^{\alpha}(n_4, n_3; a)\Delta\rho(n_4)\sum_{j=n_4}^{m}\left(\frac{\Delta x(j)}{f^{1/\alpha}(x(j+1))}\right)^{\alpha} \\ &\quad\leq -\sum_{j=n_4}^{m}\rho(j)q(4j) + h^{\alpha}(n_4, n_3; a)\Delta\rho(n_4)\left(\sum_{j=n_4}^{m}\frac{\Delta x(j)}{f^{1/\alpha}(x(j+1))}\right)^{\alpha} \\ &\quad\leq -\sum_{j=n_4}^{m}\rho(j)q(4j) + h^{\alpha}(n_4, n_3; a)\Delta\rho(n_4)\left(\int_{x(n_4)}^{x(m+1)}\frac{dy}{f^{1/\alpha}(y)}\right)^{\alpha}. \end{aligned}$$

Using (3.14) and (3.16) in the above inequality, we find that $0 < w(m+1) \to -\infty$ as $m \to \infty$, which is a contradiction and completes the proof. □

The following result is concerned with the case when

$$\sum^{\infty} q(4j) < \infty. \tag{3.19}$$

Theorem 3.5. *If in addition to conditions* (1.2), (3.14) *and* (3.19),

$$\lim_{n\to\infty}\sum_{s=n_0}^{n} h(s, n_0; a)\left(\sum_{j=s+1}^{\infty} q(4j)\right)^{1/\alpha} = \infty, \tag{3.20}$$

then equation (1.1) *is oscillatory.*

Proof. Let $\{x(n)\}$ be a nonoscillatory solution of equation (1.1), say, $x(n) > 0$ for $n \geq n_0 \geq 1$. We define $w(n)$ as in Theorem 3.1 with $\rho(n) = 1$, to obtain

$$\Delta\left(\frac{L_3 x(4n)}{f(x(n))}\right) \leq -q(4n), \quad n \geq n_3. \tag{3.21}$$

We sum (3.21) from $n+1$ to $u-1$, to get

$$0 < \frac{L_3x(4u)}{f(x(u))} \le \frac{L_3x(4n+4)}{f(x(n+1))} - \sum_{j=n+1}^{u-1} q(4j).$$

Letting $u \to \infty$ in the above inequality, we obtain

$$\frac{L_3x(4n)}{f(x(n+1))} \ge \sum_{j=n+1}^{\infty} q(4j).$$

Using (3.4) in the above inequality, we find

$$\left(\frac{\Delta x(n)}{f^{1/\alpha}(x(n+1))}\right)^{\alpha} \ge h^{\alpha}(n, n_3; a) \sum_{j=n+1}^{\infty} q(4j), \quad n \ge n_3,$$

or

$$h(n, n_3; a)\left(\sum_{j=n+1}^{\infty} q(4j)\right)^{1/\alpha} \le \frac{\Delta x(n)}{f^{1/\alpha}(x(n+1))}, \quad n \ge n_3.$$

Summing the above inequality from n_3 to n, we find

$$\sum_{s=n_3}^{n} h(s, n_3; a)\left(\sum_{j=s+1}^{\infty} q(4j)\right)^{1/\alpha} \le \sum_{i=n_3}^{n} \frac{\Delta x(i)}{f^{1/\alpha}(x(i+1))} \le \int_{x(n_3)}^{x(n+1)} \frac{dy}{f^{1/\alpha}(y)}.$$

Taking limit of both sides of the above inequality as $n \to \infty$, we arrive at the desired contradiction. This completes the proof. □

Next, we present the following comparison result.

Theorem 3.6. *Let condition* (1.2) *hold. If the equation*

$$\Delta^2 x(n) + \left(\frac{1}{a(n)} \sum_{s=n}^{\infty}\sum_{j=s}^{\infty} q(j)\right)^{1/\alpha} f^{1/\alpha}(x(n+1)) = 0 \tag{3.22}$$

is oscillatory, then all bounded solutions of equation (1.1) *are oscillatory.*

Proof. Let $\{x(n)\}$ be a nonoscillatory solution of equation (1.1), say, $x(n) > 0$ for $n \ge n_0 > 0$. It is easy to check that $x(n)$ satisfies Case (II) for $n \ge n_1 \ge n_0$. Summing equation (1.1) from $n \ge n_1$ to u and letting $u \to \infty$, we have

$$L_3x(n) \ge \left(\sum_{j=n}^{\infty} q(j)\right) f(x(n+1)) \quad \text{for} \quad n \ge n_1.$$

Once again, summing this equation from $n \ge n_1$ to u and letting $u \to \infty$, we obtain

$$-L_2x(n) \ge \left(\sum_{s=n}^{\infty}\sum_{j=s}^{\infty} q(j)\right) f(x(n+1)) \quad \text{for} \quad n \ge n_1,$$

or

$$\Delta^2 x(n) + \left(\frac{1}{a(n)}\sum_{s=n}^{\infty}\sum_{j=s}^{\infty} q(j)\right)^{1/\alpha} f^{1/\alpha}(x(n+1)) \le 0 \quad \text{for} \quad n \ge n_1.$$

Applying known result (see [2,9], also Theorem 5.1 (below)), we see that equation (3.22) is nonoscillatory, which contradicts the hypothesis and completes the proof. □

4. Further Oscillation Criteria

In this section we shall consider difference equations of type (1.1) with delay, i.e.,

$$L_4 x(n) + q(n) f(x[n-\tau+1]) = 0, \tag{4.1}$$

where $\tau \ge 0$ is a real number. Our main goal is to establish some oscillation criteria for equation (4.1) and some necessary and sufficient conditions for the equation

$$L_4 x(n) + q(n) x^{\beta}[n-\tau+1] = 0, \tag{4.2}$$

where β is the ratio of two positive odd integers.

Theorem 4.1. *Let condition* (1.2) *hold and assume that f satisfies*

$$-f(-xy) \ge f(xy) \ge f(x)f(y) \quad \text{for} \quad xy > 0. \tag{4.3}$$

If for all large $n \ge n_0 + \tau$ the first order difference equation

$$\Delta y(n) + q(n) f(H(n-\tau, n_0; a)) f\left(y^{1/\alpha}[n-\tau]\right) = 0 \tag{4.4}$$

is oscillatory, then equation (4.1) *is oscillatory.*

Proof. Let $\{x(n)\}$ be a nonoscillatory solution of equation (4.1), say, $x(n) > 0$ for $n \ge n_0 \ge 0$. By Lemma 2.1, there exists $n_1 \ge n_0 + \tau$ so large that

$$x[n-\tau+1] \ge H(n-\tau, n_0; a) L_3^{1/\alpha} x[n-\tau] \quad \text{for} \quad n \ge n_1. \tag{4.5}$$

Using (4.5) and (4.3) in equation (4.1), we have

$$\begin{aligned} -\Delta L_3 x(n) &\ge q(n) f(x[n-\tau]) \\ &\ge q(n) f(H(n-\tau, n_0; a)) f\left(L_3^{1/\alpha} x[n-\tau]\right), \quad n \ge n_1. \end{aligned}$$

Substituting $u(n)$ for $L_3 x(n)$, $n \ge n_1$, we get

$$-\Delta u(n) \ge q(n) f(H(n-\tau, n_0; a)) f\left(u^{1/\alpha}[n-\tau]\right) \quad \text{for} \quad n \ge n_1. \tag{4.6}$$

Summing (4.6) from $n \ge n_1$ to $k \ge n$ and letting $k \to \infty$, we obtain

$$u(n) \ge \sum_{s=n}^{\infty} q(s) f(H(s-\tau, n_0; a)) f\left(u^{1/\alpha}[s-\tau]\right).$$

Now as in [2] it is easy to conclude that there exists a positive solution $\{y(n)\}$ of equation (4.4) with $\limsup_{n\to\infty} y(n) = 0$, which contradicts the hypothesis and completes the proof. □

The following corollaries are immediate.

Corollary 4.1. *Let condition* (1.2) *hold. If for all large* $n \geq n_0 + \tau$,

$$\liminf_{n\to\infty} \sum_{j=n-\tau}^{n-1} q(j)H^{\alpha}(j-\tau, n_0; a) > \left(\frac{\tau}{\tau+1}\right)^{\tau+1}, \quad \textit{when} \quad \alpha = \beta \tag{4.7}$$

or

$$\sum^{\infty} q(j)H^{\beta}(j-\tau, n_0; a) = \infty, \quad \textit{when} \quad \beta < \alpha \tag{4.8},$$

then equation (4.2) *is oscillatory.*

Corollary 4.2. *Let condition* (1.2) *hold. If for all large* $n \geq n_0 + \tau$,

$$\sum^{\infty} q(j)H_1^{\beta}(j-\tau, n_0; a) = \infty, \quad \textit{when} \quad \beta < \alpha, \tag{4.9}$$

where

$$H_1(n, n_0; a) = \sum_{k=n_0}^{n-1} \sum_{j=n_0}^{k-1} \left(\frac{j-n_0}{a(j)}\right)^{1/\alpha},$$

then all unbounded solutions of equation (4.2) *are oscillatory.*

Proof. Let $\{x(n)\}$ be an unbounded nonoscillatory solution of equation (4.2), say, $x(n) > 0$ for $n \geq n_0$. It is easy to see that $\{x(n)\}$ satisfies Case (I), and so,

$$x(n) \geq H_1(n, n_0; a)L_3^{1/\alpha}x(n), \quad n \geq n_1 \geq n_0.$$

The rest of the proof is easy and hence omitted. □

For the oscillation of all bounded solutions of equation (4.1) we present the following result.

Theorem 4.2. *Let condition* (2.1) *hold. If*

$$\sum_{s_3=n_0}^{\infty} \sum_{s_2=s_3}^{\infty} \left(\frac{1}{a(s_2)} \sum_{s_1=s_2}^{\infty} \sum_{s=s_1}^{\infty} q(s)\right)^{1/\alpha} = \infty, \tag{4.10}$$

then all bounded solutions of equation (4.1) *are oscillatory.*

Proof. Let $\{x(n)\}$ be a bounded nonoscillatory solution of equation (4.1), say, $x(n) > 0$ for $n \geq n_0 \geq 0$. Clearly, $x(n)$ satisfies (II) for $n \geq n_1$ for some $n_1 \geq n_0$. Now, there exist a constant $c > 0$ and an $n_2 \geq n_1$ such that

$$\frac{c}{2} \leq x[n-\tau+1] \leq c \quad \text{for} \quad n \geq n_2. \tag{4.11}$$

Using (4.11) in equation (4.1), we have

$$L_4x(n) + q(n)f(c/2) \leq 0 \quad \text{for} \quad n \geq n_2.$$

Summing this inequality from $n \geq n_2$ to $u \geq n$ and letting $u \to \infty$, we have

$$L_3x(n) \geq f(c/2)\sum_{s=n}^{\infty} q(s).$$

Once again, summing the above inequality from $n \geq n_2$ to u and letting $u \to \infty$, we obtain

$$-\Delta^2 x(n) \geq f^{1/\alpha}(c/2)\left(\frac{1}{a(n)}\sum_{s_1=n}^{\infty}\sum_{s=s_1}^{\infty} q(s)\right)^{1/\alpha}.$$

Therefore, we find

$$x(n) \geq x(n_2) + f^{1/\alpha}\left(\frac{c}{2}\right)\sum_{s_3=n_2}^{\infty}\sum_{s_2=s_3}^{\infty}\left(\frac{1}{a(s_2)}\sum_{s_1=s_2}^{\infty}\sum_{s=s_1}^{\infty} q(s)\right)^{1/\alpha} \to \infty \text{ as } n \to \infty,$$

which is a contradiction and completes the proof. □

The following theorem is concerned with a necessary and sufficient condition for the oscillation of all unbounded solutions of equation (4.2) with $\beta < \alpha$.

Theorem 4.3. *Let $\beta < \alpha$ and condition (1.2) hold. All unbounded solutions of equation (4.2) are oscillatory if and only if condition (4.9) holds.*

Proof. Let $\{x(n)\}$ be an unbounded nonoscillatory solution of equation (4.2), say, $x(n) > 0$ for $n \geq n_0 \geq 0$. Clearly, $x(n)$ satisfies (I) for $n \geq n_1$ for some $n_1 \geq n_0$. The proof of the "if" part is as in Corollary 4.2 and hence omitted.

To prove the "only if" part it suffices to assume that

$$\sum^{\infty} q(j) H_1^{\beta}(j-\tau, n_0; a) < \infty, \tag{4.12}$$

where H_1 is as in Corollary 4.2 and show the existence of a nonoscillatory solution of equation (4.2). □

Let $c > 0$ be an arbitrary constant and choose $N > n_1 \geq n_0 + \tau$ sufficiently large so that

$$\sum_{j=N}^{\infty} q(j) H_1^{\beta}(j-\tau, n_0; a) < c^{1-(\beta/\alpha)}. \tag{4.13}$$

Define the set X_1 of all real sequences $\{x(n)\}$, $n \geq N$, i.e.,

$$X_1 = \{x(n) : x(n) \text{ is defined for } n \geq N\}.$$

Now define the set X by

$$X = \{x(n) \in X_1 : \ c_1 H_1(n, N; a) \leq x(n) \leq c_2 H_1(n, N; a),\ n \geq N\},$$

where $c_1 = (c/2)^{1/\alpha}$ and $c_2 = (2c)^{1/\alpha}$.

Clearly, X is a closed convex subset of the locally convex space X_1 and X is also a compact subset of X_1.

Let S be a mapping defined on X as follows: For $x \in X$,

$$(Sx)(n) = \sum_{s_3=N}^{n-1}\sum_{s_2=N}^{s_3-1}\left(\frac{1}{a(s_2)}\left[c(s_2-N) + \sum_{s_1=N}^{s_2-1}\sum_{s=s_1}^{\infty} q(s) x^{\beta}[s-\tau+1]\right]\right)^{1/\alpha} \quad \text{for} \quad n \geq N. \tag{4.14}$$

It is easy to check that S is well defined and continuous (see Theorem 16.4 [6]). It can be shown without any difficulty that S maps X into itself and $S(X)$ is relatively compact in X_1. Therefore, by the Schauder fixed point theorem, S has a fixed point x in X which satisfies

$$x(n) = \sum_{s_3=N}^{n-1}\sum_{s_2=N}^{s_2-1}\left(\frac{1}{a(s_2)}\left[c(s_2-N)+\sum_{s_1=N}^{s_2-1}\sum_{s=s_1}^{\infty} q(s)x^{\beta}[s-\tau+1]\right]\right)^{1/\alpha} \quad \text{for} \quad n \geq N.$$

Taking the difference 4-times on the above equation, we see that $x = x(n)$ is a positive solution of equation (4.2) for $n \geq N$ such that

$$\lim_{n\to\infty}\frac{x(n)}{H_1(n,N;a)} = \gamma > 0, \quad \gamma \quad \text{is a constant.}$$

Theorem 4.3 can be restated as follows:

Theorem 4.3′. *Let $\beta < \alpha$ and condition* (1.2) *hold. Equation* (4.2) *has a nonoscillatory solution $\{x(n)\}$ such that $\lim_{n\to\infty}\frac{x(n)}{H_1(n,n_0;a)}$ is a nonzero constant, $n \geq n_0$, if and only if*

$$\sum^{\infty} q(j)H_1^{\beta}(j-\tau,n_0;a) < \infty.$$

Next, we present the following necessary and sufficient condition for the oscillation of all bounded solutions of equation (4.2) with $\beta > \alpha$.

Theorem 4.4. *Let $\beta > \alpha$ and condition* (1.2) *hold. All bounded solutions of equation* (4.2) *are oscillatory if and only if condition* (4.10) *holds.*

Proof. Let $\{x(n)\}$ be a bounded nonoscillatory solution of equation (4.2), say, $x(n) > 0$ for $n \geq n_0 \geq 0$. Clearly, $x(n)$ satisfies (II) for $n \geq n_1$ for some $n_1 \geq n_0$. The proof of the "if" part is presented in Theorem 4.2 and hence omitted.

The "only if" part of the theorem is proved as follows: Let $c > 0$ be a given arbitrary constant, and choose a large $N \geq n_1$ such that

$$\sum_{s_3=N}^{\infty}\sum_{s_2=s_3}^{\infty}\left(\frac{1}{a(s_2)}\sum_{s_1=s_2}^{\infty}\sum_{s=s_1}^{\infty} q(s)\right)^{1/\alpha} \leq \frac{1}{2}c^{1-(\beta/\alpha)}.$$

We introduce the Banach space ℓ^{∞} of all bounded, real sequences $\{x(n)\}$ ($n \geq N$) with the norm $\|x\| = \sup_n |x(n)|$. We define a bounded convex and closed subset $\mathcal{B}$ of ℓ^{∞} as

$$\mathcal{B} = \{x \in \ell^{\infty} : \ c/2 \leq x[n-\tau+1] \leq c, \ \ n \geq N\}.$$

Next, let T be a mapping defined on $\mathcal{B}$ as follows: For $x = x(n) \in \mathcal{B}$,

$$(Tx)(n) = c - \sum_{s_3=N}^{n-1}\sum_{s_2=s_3}^{\infty}\left(\frac{1}{a(s_2)}\sum_{s_1=s_2}^{\infty}\sum_{s=s_1}^{\infty} q(s)x^{\beta}[s-\tau+1]\right)^{1/\alpha}.$$

It is easy to check that T maps $\mathcal{B}$ into itself and T is a continuous mapping. Also, $T(\mathcal{B})$ is relatively compact in ℓ^∞. Therefore, by Schauder fixed point theorem, there exists an element $x \in \mathcal{B}$ such that $x = Tx$. It is clear that the fixed point $x = x(n)$ gives a positive solution of equation (4.2) for $n \geq N$ such that $\lim_{n\to\infty} x(n) = c$ (for details, see Theorem 3.1 in [12] and Theorem 16.5 in [6]). □

Once again, we restate Theorem 4.4 as follows:

Theorem 4.4′. *Let $\beta > \alpha$ and condition (1.2) hold. Equation (4.2) has a nonoscillatory solution $\{x(n)\}$ such that $\lim_{n\to\infty} x(n)$ is a nonzero constant if and only if*

$$\sum^{\infty}\sum_{s_2=s_3}^{\infty}\left(\frac{1}{a(s_2)}\sum_{s_1=s_2}^{\infty}\sum_{s=s_1}^{\infty}q(s)\right)^{1/\alpha} < \infty.$$

Next, we consider equation (4.1) when $\tau < 0$, i.e., we consider the advanced difference equation

$$L_4x(n) + q(n)f(x[n+\tau+1]) = 0. \tag{4.15}$$

Theorem 4.5. *Let condition (1.2) hold. Then, equation (4.15) is oscillatory if either one of the following conditions holds*

(I$_1$). $\tau > 1$,

$$\frac{f^{1/\alpha}(x)}{x} \geq k > 0 \quad \text{for} \quad x \neq 0 \quad \text{and} \quad k \text{ is a constant} \tag{4.16}$$

and for all large $n > n_0$,

$$\liminf_{n\to\infty}\sum_{i=n+1}^{n+\tau-1} h(i, n_0; a)\left(\sum_{s=4i}^{\infty}q(s)\right)^{1/\alpha} > \frac{1}{k}\left(\frac{\tau-1}{\tau}\right)^{\tau}, \tag{4.17}$$

or, for all large $n \geq n_0$,

$$\limsup_{n\to\infty} H(n, n_0; a)\left(\sum_{s=n}^{\infty}q(s)\right)^{1/\alpha} > \frac{1}{k}. \tag{4.18}$$

(I$_2$). $\tau \geq 1$, *condition (3.14) holds and for all large $n \geq n_0$,*

$$\sum^{\infty} h(i, n_0; a)\left(\sum_{s=i}^{\infty}q(s)\right)^{1/\alpha} = \infty. \tag{4.19}$$

Proof. Let $\{x(n)\}$ be an eventually positive solution of equation (4.15), say, $x(n) > 0$ for $n \geq n_0$. By Lemma 2.1, there exists an $n_1 \geq n_0$ such that

$$\Delta x(n) \geq h(n, n_0; a)L_3^{1/\alpha}x(4n) \quad \text{for} \quad n \geq n_1 \tag{4.20}$$

and

$$x(n) \geq H(n, n_0; a)L_3^{1/\alpha}x(n) \quad \text{for} \quad n \geq n_1. \tag{4.21}$$

Summing equation (4.15) from $n \geq n_1$ to $u \geq n$ and letting $u \to \infty$, we get

$$L_3 x(n) \geq \left(\sum_{s=n}^{\infty} q(s) \right) f(x[n+\tau]). \tag{4.22}$$

Using (4.22) in (4.20) and the fact that $\Delta x(n) > 0$ for $n \geq n_1$, we have

$$\begin{aligned} \Delta x(n) &\geq h(n, n_0; a) L_3^{1/\alpha} x(4n) \\ &\geq h(n, n_0; a) \left(\sum_{s=4n}^{\infty} q(s) \right)^{1/\alpha} f^{1/\alpha}(x[n+\tau]) \quad \text{for} \quad n \geq n_1. \end{aligned} \tag{4.23}$$

Using condition (4.16) in (4.23), we obtain

$$\Delta x(n) \geq k h(n, n_0; a) \left(\sum_{s=4n}^{\infty} q(s) \right)^{1/\alpha} x[n+\tau] \quad \text{for} \quad n \geq n_1. \tag{4.24}$$

But, in view of Theorem 3′ in [11] and condition (4.17), inequality (4.24) has no eventually positive solution, which is a contradiction.

Next, using (4.21) in (4.22), we obtain

$$x(n) \geq H(n, n_0; a) L_3^{1/\alpha} x(n) \geq H(n, n_0; a) \left(\sum_{s=n}^{\infty} q(s) \right)^{1/\alpha} f^{1/\alpha}(x(n)),$$

or

$$\frac{x(n)}{f^{1/\alpha}(x(n))} \geq H(n, n_0; a) \left(\sum_{s=n}^{\infty} q(s) \right)^{1/\alpha} \quad \text{for} \quad n \geq n_1. \tag{4.25}$$

Taking lim sup of both sides of the above inequality, we arrive at the desired contradiction.

(I_2). Using the fact that $\Delta x(n) > 0$ for $n \geq n_1$ in (4.23), we get

$$\Delta x(n) \geq h(n, n_0; a) \left(\sum_{s=n}^{\infty} q(s) \right)^{1/\alpha} f^{1/\alpha}(x(n+1)) \quad \text{for} \quad n \geq n_1. \tag{4.26}$$

Summing inequality (4.26) from n_1 to n, we find

$$\sum_{s=n_1}^{n} h(s, n_0; a) \left(\sum_{j=s}^{\infty} q(j) \right)^{1/\alpha} \leq \sum_{s=n_1}^{n} \frac{\Delta x(s)}{f^{1/\alpha}(x(s+1))} \leq \int_{x(n_1)}^{x(n+1)} \frac{du}{f^{1/\alpha}(u)} < \infty$$

as $n \to \infty$, which contradicts condition (4.19). □

The following corollary is immediate.

Corollary 4.3. *Let condition* (1.2) *hold. If* $\lim_{x \to \infty} x f^{-1/\alpha}(x) = c$, *where* c *is a nonnegative constant, and*

$$\limsup_{n \to \infty} H(n, n_0; a) \left(\sum_{s=n}^{\infty} q(s) \right)^{1/\alpha} > c,$$

then all unbounded solutions of equation (4.1) *are oscillatory.*

Proof. It follows from the inequality (4.25). □

Remark 4.1. We may note that Theorems 4.3 and 4.4 can be extended to equation (4.1). Such investigations are left to the reader.

5. Comparison and Extensions

Here, we shall give a comparison theorem which is useful to extend the obtained results to neutral equations of the form

$$L_4(x(n) + px[n - \delta\sigma]) + q(n)f(x[n - \tau + 1]) = 0, \qquad (5.1;\delta)$$

where L_4 is defined in (1.3), q and f are as in equation (1.1), $\delta = \pm 1$, p, τ and σ are nonnegative constants.

Now, we shall prove the following comparison theorem.

Theorem 5.1. *Let condition* (1.2) *hold. If the inequality*

$$L_4x(n) + q(n)f(x[n - \tau + 1]) \le 0 \quad (\ge 0) \qquad (5.2)$$

has an eventually positive (*negative*) *solution, then equation* (4.1) *also has eventually positive* (*negative*) *solution.*

Proof. Let $\{x(n)\}$ be an eventually positive solution of inequality (5.2). There exists an $n_0 \ge 0$ such that $x(n) > 0$ for $n \ge n_0$ and $x(n)$ satisfies either (I) or (II) for $n \ge n_0$. Summing inequality (5.2) from $n \ge n_0$ to $u \ge n$ and letting $u \to \infty$, we get

$$L_3x(n) \ge \sum_{s=n}^{\infty} q(s)f(x[s - \tau + 1]). \qquad (5.3)$$

Now, we need to distinguish the following two cases:
Case (I). $L_ix(n) > 0$ for $n \ge n_0$, $i = 0, 1, 2, 3$. Summing (5.3) from n_0 to $n - 1 \ge n_0$, we have

$$L_2x(n) \ge \sum_{s_1=n_0}^{n-1} \sum_{s=s_1}^{\infty} q(s)f(x[s - \tau + 1]),$$

or

$$\Delta^2 x(n) \ge \left(\frac{1}{a(n)} \sum_{s_1=n_0}^{n-1} \sum_{s=s_1}^{\infty} q(s)f(x[s - \tau + 1])\right)^{1/\alpha}$$

and so,

$$\begin{aligned} x(n) &\ge \sum_{s_3=n_0}^{n-1} \sum_{s_2=n_0}^{s_3-1} \left(\frac{1}{a(s_2)} \sum_{s_1=n_0}^{s_2-1} \sum_{s=s_1}^{\infty} q(s)f(x[s - \tau + 1])\right)^{1/\alpha} \\ &=: c + \Phi(n, x[n - \tau + 1]), \end{aligned} \qquad (5.4)$$

where $x(n_0) = c$.

Now, it is easy to show the existence of a positive solution to the equation

$$w(n) = c + \Phi(n, w[n - \tau + 1]) \quad \text{for} \quad n \ge n_0.$$

For this, we define the sequence $\{w_n(k)\}$, $k = 0, 1, 2, \ldots$, such that $w_0(n) = x(n)$, and

$$w_{k+1}(n) = \begin{cases} c + \Phi(n, w_k[n-\tau+1]) & \text{for } n \geq n_0, \\ c & \text{for } n \leq n_0. \end{cases} \tag{5.5}$$

Then, one can easily see that $w_k(n)$ is well-defined and

$$0 \leq w_k(n) \leq x(n), \quad c \leq w_{k+1}(n) \leq w_k(n).$$

Thus, the sequence $\{w_k(n)\}$ is positive and nonincreasing in k for each n. This means we may define $w(n) = \lim_{k\to\infty} w_k(n)$. Since $0 < w(n) \leq w_k(n) \leq x(n)$ for all $k \geq 0$ and since

$$\Phi(n, w_k[n-\tau+1]) \leq \Phi(n, x[n-\tau+1]),$$

the convergence of (5.5) is uniform with respect to k. Now, taking the limit of both sides of (5.5), we have

$$w(n) = c + \Phi(n, w[n-\tau+1]). \tag{5.6}$$

Finally, taking the differences of (5.6), we obtain

$$L_4 w(n) + q(n) f(w[n-\tau+1]) = 0.$$

Case (II). $L_0x(n) > 0$, $L_1x(n) > 0$, $L_2x(n) < 0$, $L_3x(n) > 0$ for $n \geq n_0$. Summing (5.3) from $n \geq n_0$ to u and letting $u \to \infty$, we have

$$-\Delta^2 x(n) \geq \left(\frac{1}{a(n)} \sum_{s_1=n}^{\infty} \sum_{s=s_1}^{\infty} q(s) f(x[s-\tau+1]) \right)^{1/\alpha}$$

and so

$$\begin{aligned} x(n) &\geq x(n_0) + \sum_{s_3=n_0}^{n-1} \sum_{s_3=s_2}^{\infty} \left(\frac{1}{a(s_2)} \sum_{s_1=s_2}^{\infty} \sum_{s=s_1}^{\infty} q(s) f(x[s-\tau+1]) \right)^{1/\alpha} \\ &=: c + \Psi(x, x[n-\tau+1]), \end{aligned}$$

where $x(n_0) = c$.

The rest of the proof is similar to that of Case (I) and hence omitted. □

Next, we shall employ Theorem 5.1 to extend the obtained results to the neutral difference equation $(5.1;\delta)$. In fact, we have the following comparison results.

Theorem 5.2. *Let conditions* (1.2) *and* (4.3) *hold,* $\delta = 1$ *and* $0 < p < 1$. *If the equation*

$$L_4 x(n) + q(n) f(1-p) f(x[n-\tau+1]) = 0 \tag{5.7}$$

is oscillatory, then equation $(5.1; 1)$ *is oscillatory.*

Theorem 5.3. *Let conditions* (1.2) *and* (4.3) *hold,* $\delta = -1$ *and* $p > 1$. *If the equation*

$$L_4 x(n) + q(n) f\left(\frac{p-1}{p^2}\right) f(x[n-\tau-\sigma+1]) = 0 \tag{5.8}$$

is oscillatory, then equation $(5.1;-1)$ *is oscillatory.*

Proofs of Theorems 5.2 *and* 5.3. Let $\{x(n)\}$ be a nonoscillatory solution of equation $(5.1;\delta)$, say, $x(n) > 0$ for $n \geq n_0 \geq 0$. Define

$$y(n) = x(n) + px[n - \delta\sigma], \quad n \geq n_0.$$

Then, for $n \geq n_0$,

$$L_4 y(n) + q(n) f(x[n-\tau+1]) = 0. \tag{5.9}$$

It is easy to check that there exists an $n_1 \geq n_0$ such that $\Delta y(n) > 0$ for $n \geq n_1$. Now, by using the hypotheses of Theorem 5.2, we find

$$\begin{aligned} x(n) &= y(n) - px[n-\sigma] = y(n) - p(y[n-\sigma] - px[n-2\sigma]) \\ &\geq y(n) - py[n-\sigma] \geq (1-p)y(n) \quad \text{for} \quad n \geq n_0. \end{aligned} \tag{5.10}$$

Using (5.10) and condition (4.3) in equation (5.9), we obtain

$$L_4 y(n) + q(n) f(1-p) f(y[n-\tau+1]) \leq 0 \quad \text{for} \quad n \geq n_1. \tag{5.11}$$

Next, using the hypotheses of Theorem 5.3, we find

$$\begin{aligned} x(n) &= \frac{1}{p}(y[n-\sigma] - x[n-\sigma]) = \frac{1}{p} y[n-\sigma] - \frac{1}{p^2} y[n-2\sigma] + \frac{1}{p^2} x[n-2\sigma] \\ &\geq \left(\frac{p-1}{p^2}\right) y[n-\sigma] \quad \text{for} \quad n \geq n_1. \end{aligned} \tag{5.12}$$

Using (5.12) and condition (4.3) in equation (5.9), we have

$$L_4 y(n) + q(n) f\left(\frac{p-1}{p^2}\right) f(y[n-\sigma-\tau+1]) \leq 0 \quad \text{for} \quad n \geq n_1. \tag{5.13}$$

Inequalities (5.11) and (5.13) have eventually positive solutions and so, by Theorem 5.1, equations (5.7) and (5.8) have also eventually positive solutions, which contradicts the hypotheses and completes the proofs. □

Finally, we shall extend our previous results to equation (1.1), or equation (4.1) when the function f need not be monotonic. For this the following notation and a lemma due to Mahfoud [10] will be needed.

$$\mathbb{R}_{t_0} = \begin{cases} (-\infty, -t_0] \cup [t_0, \infty) & \text{if } t_0 > 0, \\ \mathbb{R} - \{0\} & \text{if } t_0 = 0 \end{cases}$$

and

$$C_B(\mathbb{R}_{t_0}) = \{f \in C(\mathbb{R}) : f \text{ is of bounded variation on any interval } [a,b] \subseteq \mathbb{R}_{t_0}\}.$$

Lemma 5.1. *Suppose* $t_0 > 0$ *and* $f \in C(\mathbb{R}) = \{f \in C(\mathbb{R}, \mathbb{R}) : xf(x) > 0 \text{ for } x \neq 0\}$. *Then,* $f \in C_B(\mathbb{R}_{t_0})$ *if and only if* $f(x) = H(x)G(x)$ *for all* $x \in \mathbb{R}$, *where* $G : \mathbb{R}_{t_0} \to \mathbb{R}^+$ *is nondecreasing on* $(-\infty, -t_0)$ *and nonincreasing on* (t_0, ∞) *and* $H : \mathbb{R}_{t_0} \to \mathbb{R}$ *is nondecreasing on* $\mathbb{R}_{t_0}$.

Theorem 5.4. *Let condition* (1.2) *hold and assume that* $f \in C(\mathbb{R}_{t_0})$, $t_0 \geq 0$, *and let* G *and* H *be a pair of continuous components of* f *with* H *being the nondecreasing one. If, for all large* $n > n_0 + \tau$, *the equation*

$$L_4 x(n) + q(n) G(g(n-\tau+1, n_0; a)) H(x[n-\tau+1]) = 0 \tag{5.14}$$

is oscillatory, where

$$g(n, n_0; a) = \sum_{s=n_0}^{n-1} \sum_{j=n_1}^{s-1} \left(\frac{j-n_0}{a(j)} \right)^{1/\alpha}, \quad n \geq n_0 + \tau,$$

then equation (4.1) *is oscillatory.*

Proof. Let $\{x(n)\}$ be an eventually positive solution of equation (4.1), say, $x(n) > 0$ for $n \geq n_0 \geq 0$. There exist a constant $b > 0$ and an $n_1 \geq n_0$ such that

$$L_3 x(n) \leq b \quad \text{for} \quad n \geq n_1.$$

Summing the above inequality 3-times from n_1 to $n-1$, we get

$$x(n) \leq b \sum_{s=n_1}^{n-1} \sum_{j=n_1}^{s-1} \left(\frac{j-n_1}{a(j)} \right)^{1/\alpha} =: g(n, n_1; a) \quad \text{for} \quad n \geq n_1.$$

Now there exists an $n_2 \geq n_1 + \tau$ such that

$$x(n-\tau+1) \leq g(n-\tau+1, n_1; a) \quad \text{for} \quad n \geq n_2. \tag{5.15}$$

Next, since

$$\begin{aligned} f(x[n-\tau+1]) &= G(x[n-\tau+1]) H(x[n-\tau+1]) \\ &\geq G(g(n-\tau+1, n_1; a)) H(x[n-\tau+1]), \quad n \geq n_2, \end{aligned}$$

it follows that

$$L_4 x(n) + q(n) G(g(n-\tau+1, n_1; a)) H(x[n-\tau+1]) \leq 0 \quad \text{for} \quad n \geq n_2.$$

By applying Theorem 5.1, we arrive at the desired contradiction. □

Remark 5.1. The results of this paper are presented in a form which can be easily extended to higher order nonlinear difference equations of the form

$$\Delta^m \left(a(n) (\Delta^m x(n))^\alpha \right) + \delta q(n) f(x[n-\tau+1]) = 0 \tag{5.16; δ}$$

and the forced equation

$$\Delta^m \left(a(n) (\Delta^m x(n))^\alpha \right) + \delta q(n) f(x[n-\tau+1]) = e(n), \tag{5.17; δ}$$

where $m \geq 1$ is an integer, $\delta = \pm 1$, $\{e(n)\}$ is a sequence of real numbers f, $q(n)$, τ and α are as in equation (4.1). The details and appropriate investigations are left to the reader.

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