.

On Well-posedness of Impulsive Problems for Nonlinear Parabolic Equations *

Weinian Zhang[§], **Ravi P. Agarwal**[†] and **Elvan Akın–Bohner**[&]

[§]Department of Mathematics, Sichuan University

Chengdu 610064, P.R. China

wnzhang@scu.edu.cn

[†]Department of Mathematical Sciences

Florida Institute of Technology

150 W University Boulevard Melbourne, FL 32901

agarwal@fit.edu

[&]Department of Mathematics-Statistics,

University of Nebraska-Lincoln

eakin@math.unl.edu

ABSTRACT

In this paper we give weaker conditions for well-posedness of impulsive problems for nonlinear parabolic equations, including existence, uniqueness and continuous dependence, even in case their corresponding Cauchy problems do not have global solutions. With some concrete examples of PDEs we show when their impulsive problems are well-posed and how the well-posedness is related to life-span of PDEs. We also indicate that some assumptions in [6] and [9] are unnecessary.

Keywords and phrases: impulsive problem, parabolic equations, blow-up, well-posed

AMS(1991) Subject Classification: 35K70, 4G20

*Supported by NNSFC(China) Grant and China Education Ministry Research Grants

1 Introduction

An impulsive problem of differential equation can be regarded as a combination of a Cauchy problem and short-term perturbations, the durations of which are negligible in comparison with the duration of the whole process. Many phenomena subjected to abrupt changes can be modelled as impulsive problems of differential equations (see [7] and [10]). Recently, impulsive problems for evolution equations are getting more attractive (see [1, 2, 3, 6, 9]). In 1997, Rogovchenko in [9] gave a survey to some basic results on impulsive parabolic equations with sectorial linear operators, where existence and uniqueness of bounded solutions were studied with successive approximation but a very restrictive condition was imposed on the Green's functions. Later, Liu in [6] considered relations between classical solutions and mild solutions of impulsive problems for parabolic equations with more general unbounded linear operators. However, assumptions in [6] are enough to guarantee the corresponding Cauchy problems (without impulsive conditions) have solutions globally.

The existence of global solutions for PDEs usually requires stronger conditions but, for well-posedness of impulsive problems, it may not be necessary to demand conditions so stronger. In fact, an impulsive problem is expected to be well-posed on a large interval of time even if its corresponding Cauchy problem blows up in a finite period. This idea motivates us to find more reasonable conditions for the well-posedness in this paper.

In this paper we give weaker conditions for well-posedness of impulsive problems for nonlinear parabolic equations, including existence, uniqueness and continuous dependence. With some concrete examples of PDEs we show when the impulsive problems of PDEs are well-posed and how the well-posedness is related to life-span of PDEs. We also indicate that some assumptions in [6] and [9] are unnecessary. To avoid confusion with the Laplacian operator, the difference of a function $u(t)$ is denoted by $\delta u(t)$ instead of $\Delta u(t)$.

2 Basic Results

Let X be a Banach space with the norm $\|\cdot\|$, $\mathcal{L}(X)$ denote the space of all bounded linear operators on X and

$PC([0,T_0],X) = \{u : [0,T_0] \mapsto X | u(t)$ is continuous at $t \neq t_i$, left continuous at

$t = t_i$, and the right limit $u(t_i^+)$ exists for $i = 1, 2, ..., p\}$,
where $0 < t_1 < t_2 < ... < t_p < T_0 < +\infty$, $u(t_i^+) := \lim_{t>t_i, \to t_i} u(t)$ and $u(t_i^-) := \lim_{t<t_i, \to t_i} u(t)$, which is a Banach space with the norm $\|u\|_{PC} = \sup_{t\in[0,T_0]} \|u(t)\|$, as in [4].

We discuss solutions $u(t)$ in $PC([0, T_0], X)$ for the impulsive problem of the evolution equation

$$\frac{d}{dt}u + Au = f(t, u), \qquad 0 < t < T_0,\ t \neq t_i, \tag{2.1}$$

$$u(0) = u_0 \tag{2.2}$$

$$\delta u(t_i) = I_i u(t_i), \qquad i = 1, 2, ..., p, \tag{2.3}$$

where $u(t) \in X$ and $\delta u(t_i) = u(t_i^+) - u(t_i^-)$.

As in [9], we also assume that
(H1) A is a sectorial operator in X with the sector $S_{a,\eta} := \{\lambda \in \mathbf{C} : \eta \leq |\arg(\lambda - a)| \leq \pi,\ \lambda \neq a\}$ in the resolvent set of A, where a, η are definite reals and $\eta \in (0, \pi/2)$;
(H2) $f : [0, T_0] \times X^\alpha \to X$ is continuous and Lipschitzian in $u \in X^\alpha$ with Lipschitzian constant $K > 0$, where $0 \leq \alpha \leq 1$ and X^α denotes the fractional power subspace of X;
(H3) all $I_i : X^\alpha \to X^\alpha$, $i = 1, 2, ..., p$, are continuous operators.

Recall that a linear operator A on a Banach space X is *sectorial* if A is a closed, densely defined operator such that, for some $\varphi \in (0, \pi/2)$ and some $\mu \geq 1$ and real a, the sector $S_{a,\varphi} := \{\lambda \in \mathbf{C} :\ \varphi \leq |\arg(\lambda - a)| \leq \pi, \lambda \neq a\}$ is in the resolvent set $\rho(A)$ of A, and $\|(\lambda - A)^{-1}\| \leq \frac{\mu}{|\lambda - a|}$ for all $\lambda \in S_{a,\varphi}$. Further, for any sectorial operator A there is a real number a such that $\mathrm{Re}\sigma(A + aI) > 0$. With $A_1 := A + aI$ (where I is the identity operator) we can define the *fractional power operator* A_1^α of A_1 for $0 \leq \alpha \leq 1$. Then we define the *fractional power space* X^α to be the domain $D(A_1^\alpha)$ of A_1^α, and the graph norm $\|x\|_\alpha := \|A_1^\alpha x\|$, $x \in X^\alpha$. Here we can take $A_1 := A - (a - |a|/2)I$ without loss of generality, where a is given in (H1). The concepts of sectorial operator and fractional power subspace can be found in [5]. Different choices of a give equivalent spaces X^α and equivalent norms on that space (Theorem 1.4.6 of Henry [5]), and X^α is a Banach space with the graph norm. Clearly, $-A$ generates an analytic semigroup $S(t) = e^{-tA}, t \geq 0$, on X and

$$\|e^{-At}\|_{\mathcal{L}(X)} \leq Ce^{-at}, \quad \|A_1^\alpha e^{-At}\|_{\mathcal{L}(X)} \leq Ct^{-\alpha}e^{-at} \tag{2.4}$$

for some constant $C > 0$, by Theorems 1.3.4 and 1.4.3 of [5].

In comparison with **(H3)**, Theorem 2.1 in [6] uses another stronger assumption, that all $I_i : X^\alpha \to X^\alpha$ are Lipschitzian with Lipschitz constants $k_i > 0$,

$i = 1, 2, ..., p$, to give existence and uniqueness of mild solutions (roughly speaking, the solutions of the equivalent integral equation, which are integrable in t) when $\max_{t\in[0,T_0]} \|S(t)\|_{\mathcal{L}(X)}(KT_0 + \sum_{i=1}^{p} k_i) < 1$. Actually, for a given equation this requires both $T_0 > 0$ and $k_i, i = 1, 2, ..., p$, are very small, i.e., the described impulsive phenomenon can only occur transiently and the impulses are almost constant, but most of practical models are not like this. In the following we will give a stronger result with a much weaker condition.

For convenience, we take notations $t_0 = 0$ and $t_{p+1} = T_0$. Let $\gamma = \max_{1\le i\le p+1} |t_i - t_{i-1}|$, which is called the *maximum pace* of impulsion.

Theorem 1 *Suppose* **(H1-H3)** *hold. If*

$$\frac{KC}{1-\alpha}\max\{e^{-a\gamma}, 1\}(\gamma^{1-\alpha} + |a|\gamma^{2-\alpha}) < 1 \tag{2.5}$$

then for any $u_0 \in X^\alpha$ the impulsive problem (2.1–2.3) has a unique solution $u(t)$ on $[0, T_0]$, which satisfies

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)} f(s, u(s))\, ds + \sum_{0<t_i<t} e^{-A(t-t_i)} I_i(u(t_i)), \tag{2.6}$$

for $t \in [0, T_0]$.

Theorem 2 *Under the same conditions as in Theorem 1, the solution of the problem (2.1–2.3) is continuously dependent on the initial data.*

Remark 1 *Theorem 1 and 2 indicate that with the basic assumptions* **(H1–H3)** *the impulsive problem (2.1–2.3) is well-posed for all $u_0 \in X^\alpha$ if (2.5) holds.*

Remark 2 *A slightly stronger condition that*

$$KC(1 + |a|)\frac{e^{|a|\gamma}\gamma^{1-\alpha}}{1-\alpha} < 1 \tag{2.7}$$

would be more intuitive and easier to check. It is still much weaker than that in [6]. In fact, this condition is satisfied when $\gamma > 0$ small enough, so the whole interval $[0, T_0]$ need not to be small. It only requires that the paces of impulsion are small. That means, even if the corresponding Cauchy problem blows up in $[0, T_0]$, the impulsive problem can be well-posed when the impulsive condition is given appropriately.

Remark 3 *For well-posedness, it suffices to consider the maximum pace of impulsion to be* $\gamma = \max_{1\le i\le p} |t_i - t_{i-1}|$*. In fact, after* t_p *there is no influence of impulses and existence of solutions on the final interval* $[t_p, T_0]$ *does not affect well-posedness of impulsive problems.*

For example,

$$\begin{cases} u_t = \Delta u + \sin u & 0 < x < \pi,\ t \in (0,\infty), \\ u(0,t) = u(\pi,t) & t \in (0,\infty), \\ u(x,0) = u_0(x) & 0 < x < \pi, \\ \delta u(t_i) = I_i u(t_i), & t_i \in (0,\infty),\ i = 1,2,...,p, \end{cases} \tag{2.8}$$

where Δ is the Laplacian operator. Take $X := L^2(0,\pi)$, $A := -d^2/dx^2$ with domain $H^2(0,\pi) \cap H_0^1(0,\pi)$, and $f(t,u) = \sin u$. Clearly, the spectrum $\sigma(A) = \{\lambda_n = n^2 : n = 1,2,...\}$. It is not hard to see

$$\begin{aligned} \|A_1^\alpha e^{-At}\|_{\mathcal{L}(X)} &\le \begin{cases} (te/\alpha)^{-\alpha}, & 0 < t \le \alpha/\lambda_1, \\ \lambda_1^\alpha e^{-\lambda_1 t}, & t \ge \alpha/\lambda_1, \end{cases} \\ &\le (\frac{2\alpha}{e})^\alpha t^{-\alpha} e^{-\frac{\lambda_1}{2}t}, \quad \forall t > 0. \end{aligned}$$

In particular, in $X^{1/2} = D(A^{1/2}) = H_0^1(0,\pi)$, the condition (2.7) is equivalent to $\gamma e^\gamma < e/9$. Simple computation shows that the problem (2.8) is well-posed if the pace $\gamma < 1/5$.

Those modified equations with cut-off nonlinearity in a compact subset when we discuss invariant manifolds also have nonlinear terms with global Lipschitzian condition. In the same way we can also give an estimate of maximum pace for well-posedness.

As previous we see well-posedness of impulsive problems is related to blow-up and life span problem of the corresponding Cauchy problems. Consider the Cauchy problem

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u & \text{in } \mathbf{R}^N \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbf{R}^N, \end{cases} \tag{2.9}$$

where $p > 1$. Mizoguchi and Yanagida in [8] gives an estimate of lower bound of life-span in its Theorem 2.4, that is, for some initial data $u_0(x) = \lambda\phi \in W^{1,\infty}(\mathbf{R}^N)$, $\lambda > 0$ is a constant, the life-span

$$T(\lambda) \ge \lambda^{-(\frac{1}{p-1} - \frac{\beta}{2(1-\mu)})^{-1} + \varepsilon}$$

where β, μ, ε are positive constant. Obviously, its impulsive problem with the condition

$$\delta u(t_1) = I_1 u(t_1), \quad 0 < t_1 \le \lambda^{-(\frac{1}{p-1} - \frac{\beta}{2(1-\mu)})^{-1} + \varepsilon},$$

is well-posed.

3 Proof of Existence and Uniqueness

For the Cauchy problem of (2.1) with $u(\tau) = \phi$, $\tau \in [0, T_0)$ and $\phi \in X^\alpha$, by Lemma 3.3.2 in [5], we can consider equivalently the integral equation

$$u(t) = e^{-A(t-\tau)}\phi + \int_\tau^t e^{-A(t-s)} f(s, u(s))\, ds. \tag{3.10}$$

Let W consist of all continuous functions $u : [\tau, \tau + \gamma] \to X^\alpha$. It is a complete metric space with the usual sup-norm $\|u\|_* = \sup_{\tau \le t \le \tau+\gamma} \|u(t)\|_\alpha$. Define a mapping $\mathcal{F} : W \to W$ by

$$\mathcal{F}u(t) = e^{-A(t-\tau)}\phi + \int_\tau^t e^{-A(t-s)} f(s, u(s))\, ds. \tag{3.11}$$

It is clearly true that $\mathcal{F}(W) \subset W$. Thus, for $u, v \in W$ and $\tau \le t \le \tau + \gamma$,

$$\begin{aligned}
\|\mathcal{F}u(t) - \mathcal{F}v(t)\|_\alpha &\le \int_\tau^t \|A_1^\alpha e^{-A(t-s)}\|_{\mathcal{L}(X)} \cdot \|f(s, u(s)) - f(s, v(s))\|\, ds \\
&\le KC \int_\tau^t (t-s)^{-\alpha} e^{-a(t-s)}\, ds \|u - v\|_* \\
&= KC \left(\frac{(t-\tau)^{1-\alpha}}{1-\alpha} e^{-a(t-\tau)} + \frac{a}{1-\alpha} \int_\tau^t (t-s)^{1-\alpha} e^{-a(t-s)}\, ds \right) \|u - v\|_* \\
&\le \frac{KC}{1-\alpha} \max\{e^{-a\gamma}, 1\} (\gamma^{1-\alpha} + |a|\gamma^{2-\alpha}) \|u - v\|_*,
\end{aligned} \tag{3.12}$$

where integration by parts and inequalities in (2.4) are applied. By (2.5), $\mathcal{F}$ is a contraction of W into W. This implies the following result.

Lemma 1 *Under the same conditions as in Theorem 1, the Cauchy problem of (2.1) with $u(\tau) = \phi$, $\tau \in [0, T_0)$ and $\phi \in X^\alpha$, has a unique solution $u \in C([\tau, \tau + \gamma], X^\alpha)$.*

In the special case where $\tau = 0$ and $\phi = u_0 \in X^\alpha$, by Lemma 1 we obtain a unique solution $u_1 \in C([0, \gamma], X^\alpha)$, which satisfies (3.10). Since $0 \le t_1 \le \gamma$, by continuity,

$$u_1(t_1) = e^{-At_1} u_0 + \int_0^{t_1} e^{-A(t_1-s)} f(s, u_1(s))\, ds \tag{3.13}$$

is well defined in X^α. Clearly $u_1(t_1) + I_1(u_1(t_1)) \in X^\alpha$. Taking $\tau = t_1$ and $\phi = u_1(t_1) + I_1(u_1(t_1))$, by Lemma 1 we also obtain a unique solution $u_2 \in C([t_1, t_1 + \gamma], X^\alpha)$, which also satisfies an integral equation of the form (3.10), i.e.,

$$u_2(t) = e^{-A(t-t_1)}[u_1(t_1) + I_1(u_1(t_1))] + \int_{t_1}^t e^{-A(t-s)} f(s, u_2(s))\, ds \tag{3.14}$$

on the interval $t \in [t_1, t_1 + \gamma]$, which obviously contains t_2 by the definition of γ. Recursively, for $i = 3, ..., p+1$ we can obtain a solution $u_i \in C([t_{i-1}, t_{i-1} + \gamma], X^\alpha)$, which also satisfies

$$\begin{aligned} u_i(t) = e^{-A(t-t_{i-1})}&[u_{i-1}(t_{i-1}) + I_{i-1}(u_{i-1}(t_{i-1}))] \\ &+ \int_{t_{i-1}}^{t} e^{-A(t-s)} f(s, u_i(s))\, ds \end{aligned} \tag{3.15}$$

on the interval $t \in [t_{i-1}, t_{i-1} + \gamma]$, which obviously contains t_i. Define

$$u(t) = \begin{cases} u_0, & t = 0, \\ u_i(t), & t_{i-1} < t \le t_i, \quad \forall i = 1, 2, ..., p+1. \end{cases} \tag{3.16}$$

Obviously, the constructed function $u(t)$ is the unique solution of the impulsive problem (2.1–2.3).

Finally, (2.6) can be proved by induction. In fact, it holds for $t \in (0, t_1]$. Assume that (2.6) holds for $t \in (t_{i-1}, t_i]$. For $t \in (t_i, t_{i+1}]$, from (3.16) and (3.15) we have

$$\begin{aligned} u(t) &= u_{i+1}(t) = e^{-A(t-t_i)}[u_i(t_i) + I_i(u_i(t_i))] + \int_{t_i}^{t} e^{-A(t-s)} f(s, u_{i+1}(s))\, ds \\ &= e^{-A(t-t_i)} u(t_i) + \int_{t_i}^{t} e^{-A(t-s)} f(s, u_{i+1}(s))\, ds + e^{-A(t-t_i)} I_i(u_i(t_i)) \\ &= e^{-A(t-t_i)}\{e^{-At_i} u_0 + \int_0^{t_i} e^{-A(t_i-s)} f(s, u(s))\, ds + \sum_{0<t_j<t_i} e^{-A(t_i-t_j)} I_j(u(t_j))\} \\ &\qquad\qquad + \int_{t_i}^{t} e^{-A(t-s)} f(s, u_{i+1}(s))\, ds + e^{-A(t-t_i)} I_i(u_i(t_i)) \\ &= e^{-At} u_0 + \int_0^{t_i} e^{-A(t-s)} f(s, u(s))\, ds + \sum_{0<t_j<t_i} e^{-A(t-t_j)} I_j(u(t_j)) \\ &\qquad + \int_{t_i}^{t} e^{-A(t-s)} f(s, u(s))\, ds + e^{-A(t-t_i)} I_i(u(t_i)) \\ &= e^{-At} u_0 + \int_0^{t} e^{-A(t-s)} f(s, u(s))\, ds + \sum_{0<t_j<t} e^{-A(t-t_j)} I_j(u(t_j)). \end{aligned} \tag{3.17}$$

The proof is complete. $\square$

4 Proof of Continuous Dependence

As proved in Theorem 1, for $u_0, v_0 \in X^\alpha$ and $t \in [0, t_1]$ we have

$$\|u(t) - v(t)\|_\alpha \le \|e^{-At}\|_{\mathcal{L}(X)} \|u_0 - v_0\|_\alpha$$

$$
\begin{aligned}
&+\int_0^t \|A_1^\alpha e^{-A(t-s)}\|_{\mathcal{L}(X)}\cdot\|f(s,u(s))-f(s,v(s))\|\,ds\\
\le\ & Ce^{-at}\|u_0-v_0\|_\alpha\\
&+KC\int_0^t (t-s)^{-\alpha}e^{-a(t-s)}\|u(s)-v(s)\|_\alpha\,ds\\
\le\ & Ce^{-at}\|u_0-v_0\|_\alpha\\
&+KC\int_0^t (t-s)^{-\alpha}e^{-a(t-s)}\,ds\sup_{0\le s\le t_1}\|u(s)-v(s)\|_\alpha\\
\le\ & Ce^{|a|\gamma}\|u_0-v_0\|_\alpha+\Psi\sup_{0\le s\le t_1}\|u(s)-v(s)\|_\alpha \qquad (4.18)
\end{aligned}
$$

where $\Psi=\frac{KC}{1-\alpha}\max\{e^{-a\gamma},1\}(\gamma^{1-\alpha}+|a|\gamma^{2-\alpha})<1$ by (2.5). Hence

$$\sup_{0\le t\le t_1}\|u(t)-v(t)\|_\alpha\le\frac{Ce^{|a|\gamma}}{1-\Psi}\|u_0-v_0\|_\alpha. \tag{4.19}$$

Similarly, by left continuity of functions in $PC([0,T_0],X)$ we can also prove by induction that

$$\sup_{t_i<t\le t_{i+1}}\|u(t)-v(t)\|_\alpha\le\frac{Ce^{|a|\gamma}}{1-\Psi}\|u(t_i^+)-v(t_i^+)\|_\alpha,\quad i=1,2,...,p, \tag{4.20}$$

that is,

$$\sup_{t_i<t\le t_{i+1}}\|u(t)-v(t)\|_\alpha\le\frac{Ce^{|a|\gamma}}{1-\Psi}(\|u(t_i)-v(t_i)\|_\alpha+\|I_i(u(t_i))-I_i(v(t_i))\|_\alpha), \tag{4.21}$$

where $i=1,2,...,p$.

For arbitrarily given $\varepsilon>0$, by continuity of I_p, there exists a constant $\rho_p>0$ such that

$$\|I_p(u(t_p))-I_p(v(t_p))\|_\alpha<\frac{1}{2}(\frac{Ce^{|a|\gamma}}{1-\Psi})^{-1}\varepsilon$$

when $\|u(t_p)-v(t_p)\|_\alpha<\rho_p$. Thus

$$\sup_{t_p<t\le t_{p+1}}\|u(t)-v(t)\|_\alpha\le\varepsilon \tag{4.22}$$

when

$$\|u(t_p)-v(t_p)\|_\alpha<\min\{\frac{1}{2}(\frac{Ce^{|a|\gamma}}{1-\Psi})^{-1}\varepsilon,\rho_p\}. \tag{4.23}$$

Let $\varepsilon_p=\min\{\frac{1}{2}(\frac{Ce^{|a|\gamma}}{1-\Psi})^{-1}\varepsilon,\rho_p,\varepsilon\}$. Similarly, there is also a constant $\rho_{p-1}>0$ such that

$$\sup_{t_{p-1}<t\le t_p}\|u(t)-v(t)\|_\alpha<\varepsilon_p<\varepsilon \tag{4.24}$$

when

$$\|u(t_{p-1}) - v(t_{p-1})\|_\alpha < \min\{\frac{1}{2}(\frac{Ce^{|a|\gamma}}{1-\Psi})^{-1}\varepsilon_p, \rho_{p-1}\}. \tag{4.25}$$

Repeating the same procedure, we can recursively define positive $\varepsilon_{p-1}, ..., \varepsilon_2, \varepsilon_1$ and correspondingly obtain positive constants $\rho_{p-2}, ..., \rho_1, \rho_0$ such that, for $i = 1, 2, ..., p-1$,

$$\sup_{t_{i-1}<t\leq t_i} \|u(t) - v(t)\|_\alpha < \varepsilon_i < \varepsilon \tag{4.26}$$

when

$$\|u(t_{i-1}) - v(t_{i-1})\|_\alpha < \min\{\frac{1}{2}(\frac{Ce^{|a|\gamma}}{1-\Psi})^{-1}\varepsilon_i, \rho_{i-1}\}. \tag{4.27}$$

This implies that

$$\|u - v\|_{PC} \leq \max_{i=1,...,p+1} \sup_{t_{i-1}<t\leq t_i} \|u(t) - v(t)\|_\alpha < \varepsilon \tag{4.28}$$

when

$$\|u(0) - v(0)\|_\alpha < \min\{\frac{1}{2}(\frac{Ce^{|a|\gamma}}{1-\Psi})^{-1}\varepsilon_1, \rho_0\}. \tag{4.29}$$

This proves the continuous dependence on initial data. □

Given a stronger condition in **(H3)** that all $I_i : X^\alpha \to X^\alpha$, $i = 1, 2, ..., p$ are Lipschitzian continuous operators, we can prove much easier the continuous dependence with (2.6) and Gronwall's inequality.

5 More Remarks on [9]

In [9] the so-called (α, β)-*Green's function* (simply called Green's function) plays an very important role that it appears in almost every theorems in [9]. It relates to the evolution operator $K(t, \tau) : X \to X^\alpha$, defined by

$$\begin{aligned} K(t,\tau) = \;& \exp(-A(t-t_p))\{\Pi_{i=1}^{p-1}(B_{i+1}+I)\exp(-A(t_{i+1}-t_i))\} \\ & \times(B_1+I)\exp(-A(t_1-\tau)), \end{aligned} \tag{5.30}$$

where I is the identity operator, B_i is the corresponding linear operator in the linear case of the impulse $I_i u(t_i) = B_i u(t_i) + b_i$ as in (2.3) and all b_i are constant vectors, $i = 1, 2, ..., p$. It is also required in [9] that

$$\int_{-\infty}^{+\infty} \|A^\beta K(t,\tau)\|\, d\tau + \sum_{i=1}^{p} \|A^\beta K(t, t_i)\| \leq L(K) \tag{5.31}$$

for a constant $L(K) > 0$ uniformly with respect to t, where $0 \leq \beta \leq \alpha < 1$ and β, α are given for fractional power subspaces X^β, X^α. As a main result, Theorem 2.2 in [9] gives existence and uniqueness of bounded solutions under the condition that

$$(L(K)+1)N(\varrho) < 1/2, \quad (L(K)+1)M(1-(L(K)+1)N(\varrho))^{-1} \leq \varrho \tag{5.32}$$

for any suffciently small ϱ, where M is a positive constant and $N(\varrho)$ is the local Lipschitzian constant of the nonlinear terms of both the equation and the impulses as $\|u\|_\alpha \leq \varrho$ such that $N(\varrho) \to 0$ as $\varrho \to 0$.

It is worthy mentioning that the requirement (5.31) in [9] is so restrictive that a simple ODE cannot satisfy it. For example, consider the ODE

$$u' = -u \tag{5.33}$$

with a trivial impulsive condition $\delta u(t_1) = 0$ (i.e., no impulse). In this circumstance, $B_1 = ... = B_p = 0$ and $K(t,\tau) = e^{-(t-\tau)}$. For simplicity, we consider $\beta = \alpha = 0$. Obviously, in (5.31),

$$\int_{-\infty}^{+\infty} \|A^\beta K(t,\tau)\| \, d\tau = \int_{-\infty}^{+\infty} e^{-(t-\tau)} \, d\tau = +\infty. \tag{5.34}$$

In addition, suppose we study impulsive problems with (5.31). The condition (5.32) restricts the existence of solutions in a very small neighborhood of 0, so actually the result is very local.

References

[1] D. Bainov, E. Minchev and K. Nakagawa, Asymptotic behavior of solutions of impulsive semilinear parabolic equations, *Nonlinear. Anal. TMA*, **30** (1997):2725-2734.

[2] C. Chan, L. Ke and A. Vatsala, Impulsive quenching for reaction-diffusion equations, *Nonlinear. Anal. TMA*, **22** (1994):1323-1328.

[3] L.H. Erbe, H.I. Freedman, X.Z. Liu and J.H. Wu, Comparison principles for impulsive parabolic equations with applications to models of single species growth, *J. Austral. Math. Soc. Ser. B*, **32**(1991):382-400.

[4] D. Guo and X.Z. Liu, Extremal solutions of nonlinear impulsive integro-differential equations in Banach spaces, *J. Math. Anal. Appl.*, **177** (1993):538-552.

[5] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, LN in Math. Vol.**840**, Springer/Berlin, 1981.

[6] J.H. Liu, Nonlinear impulsive evolution equations, *Dynamics Contin. Discr. Impulsive Syst.*, **6** (1999):77-85.

[7] X.Z. Liu and K. Rohlf, Impulsive control of a Lotka-Volterra system, *IMA J. Math. Control Inform.*, **15**(1998):269-284.

[8] N. Mizoguchi and E. Yanagida, Blowup and life span of solutions for a semilinear parabolic equation, *SIAM J. Math. Anal.*, **29**(1998):1434-1446.

[9] Y. Rogovchenko, Impulsive evolution equations: main results and new trends, *Dynamics Contin.Discr.Impulsive Syst.*, **3** (1997):57-88.

[10] S.T. Zavalishchin and A.N. Sesekin, *Dynamic Impulsive Systems, Theory and Applications*, Mathematics and Its Applications vol.**394**, Kluwer Academic, Dordrecht, 1997.