Positive Increasing Solutions of Quasilinear Dynamic Equations

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Abstract

We consider quasilinear dynamic equations whose solutions are classified into disjoint subsets by certain integral conditions. In particular, we investigate the asymptotic behavior of all positive increasing solutions of quasilinear dynamic equations.

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Running Head. Quasilinear Dynamic Equations.

1 Introduction

In this paper, we consider a quasilinear dynamic equation

$$\left[a(t)\Phi_p(x^\Delta)\right]^\Delta = b(t)f(x^\sigma), \tag{1}$$

where a and b are real positive rd-continuous functions on a **time scale** $\mathbb{T}$ (an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$), $f : \mathbb{R} \mapsto \mathbb{R}$ is continuous with $uf(u) > 0$ for $u \neq 0$ and $\Phi_p(u) = |u|^{p-2}u$ with $p > 1$. Here, we assume that $\mathbb{T}$ is unbounded above. The set of rd-continuous functions and the set of functions that are differentiable and whose derivative is rd-continuous will be denoted in this paper by C_{rd} and $\mathrm{C}^1_{\mathrm{rd}}$, respectively. By a solution we mean a delta-differentiable function x satisfying equation (1) such that $a\Phi_p(x^\Delta) \in \mathrm{C}^1_{\mathrm{rd}}$. $x^\Delta(t)$ turns out to be the usual derivative $x'(t)$ if $\mathbb{T} = \mathbb{R}$ and the usual forward difference operator $\Delta x(t) = x(t+1) - x(t)$ if $\mathbb{T} = \mathbb{Z}$, the set of integers. Therefore, Equation (1) reduces to the quasilinear differential equation, see Cecchi, Došlá and Marini [8],

$$[a(t)\Phi_p(x')]' = b(t)f(x) \tag{2}$$

when $\mathbb{T} = \mathbb{R}$ as well as to the quasilinear difference equation, see Cecchi, Došlá and Marini [7, 9],

$$\Delta[a_k\Phi_p(\Delta x_k)] = b_k f(x_{k+1}) \tag{3}$$

when $\mathbb{T} = \mathbb{Z}$. In addition we also consider the special case of equation (1)

$$\left[a(t)\Phi_p(x^\Delta)\right]^\Delta = b(t)\Phi_q(x^\sigma) \tag{4}$$

where $q > 1$.

Such studies are essentially motivated by the dynamics of positive radial solutions of reaction-diffusion (flow through porous media, nonlinear elasticity) problems modelled by the nonlinear elliptic equation

$$-\operatorname{div}(\alpha(|\nabla u|)\nabla u) + \lambda f(u) = 0, \tag{5}$$

where $\alpha : (0,\infty) \mapsto (0,\infty)$ is continuous and such that $\delta(v) := \alpha(|v|)v$ is an odd increasing homeomorphism from $\mathbb{R}$ to $\mathbb{R}$, λ is a positive constant (the Thiele modulus) and f presents the ratio of the reaction rate at concentration u to the reaction rate at concentration unity, see Diaz [10] and Grossinho and Omari [11]. If $\alpha(|v|) = |v|^{p-2}$, then the differential operator in equation (5) is the one dimensional analogue of the p–Laplacian $\Delta_p(u) = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, and equation (5) leads to equation (2) when $\mathbb{T} = \mathbb{R}$.

Our goal is to investigate the asymptotic behavior of all positive increasing solutions of equation(1) on time scales. We have some conditions on certain integrals depending on a and b to divide solutions into several disjoint subsets. The Tychonov fixed point theorem is used as well.

The setup of this paper is as follows: In Section 2, we briefly introduce preliminary results on time scales. An introduction with applications and advances in dynamic equations are given in [4, 5]. In Section 3, we consider the existence of bounded solutions of (1) on $\mathbb{T}$. In Section 4, we discuss a comparison criterion which gives the existence of upper solutions. In Sections 5 and 6, we assume that $\mu(t)$ is delta-differentiable on $\mathbb{T}$ and the existence of unbounded solutions of equations (4) and (1) on $\mathbb{T}$ is investigated, respectively.

2 Preliminary Results

The *forward jump operator* and the *backward jump operator* on $\mathbb{T}$ are defined by $\sigma(t) := \inf\{s > t : s \in \mathbb{T}\} \in \mathbb{T}$ and $\rho(t) := \sup\{s < t : s \in \mathbb{T}\} \in \mathbb{T}$ for all $t \in \mathbb{T}$, respectively. In these definitions we put $\inf(\emptyset) = \sup \mathbb{T}$ and $\sup(\emptyset) = \inf \mathbb{T}$. If $\sigma(t) > t$, we say t is *right-scattered*, while if $\rho(t) < t$, we say t is *left-scattered.* If $\sigma(t) = t$, we say t is *right-dense*, while if $\rho(t) = t$, we say t is *left-dense*. The *graininess function* $\mu : \mathbb{T} \mapsto [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$. We define the interval $[t_0, \infty)$ in $\mathbb{T}$ by $[t_0, \infty) := \{t \in \mathbb{T} : t \geq t_0\}$. The set $\mathbb{T}^\kappa$ is derived from $\mathbb{T}$ as follows: If $\mathbb{T}$ has left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

Assume $f : \mathbb{T} \mapsto \mathbb{R}$ and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$\left|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\right| \leq \epsilon \left|\sigma(t) - s\right|$$

for all $s \in U$. We call $f^\Delta(t)$ the *delta derivative* of $f(t)$ at t.

It can be shown that if $f : \mathbb{T} \mapsto \mathbb{R}$ is continuous at $t \in \mathbb{T}$ and t is right-scattered, then

$$f^\Delta(t) = \frac{f(x(\sigma(t))) - f(t)}{\mu(t)},$$

while if t is right dense, then

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s},$$

if the limit exists. If f is differentiable at t, then

$$f^\sigma(t) = f(t) + \mu(t) f^\Delta(t), \qquad \text{where} \qquad f^\sigma = f \circ \sigma. \tag{6}$$

If $f, g : \mathbb{T} \mapsto \mathbb{R}$ are differentiable at $t \in \mathbb{T}^\kappa$, then the product and quotient rules are as follows:

$$(fg)^\Delta(t) = f^\Delta(t) g(t) + f^\sigma(t) g^\Delta(t)$$

and

$$\left(\frac{f}{g}\right)^\Delta (t) = \frac{f^\Delta(t) g(t) - f(t) g^\Delta(t)}{g(t) g^\sigma(t)} \quad \text{if} \quad g(t) g^\sigma(t) \neq 0.$$

We say $f : \mathbb{T} \mapsto \mathbb{R}$ is *rd-continuous* provided f is continuous at each right-dense point $t \in \mathbb{T}$ and whenever $t \in \mathbb{T}$ is left-dense $\lim_{s\to t^-} f(s)$ exists as a finite number.

A function $F : \mathbb{T}^\kappa \mapsto \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \mapsto \mathbb{R}$ provided $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$. Every rd-continuous function has an antiderivative. In this case we define the integral of f by

$$\int_a^t f(s)\Delta s = F(t) - F(a) \quad \text{for} \quad t \in \mathbb{T}.$$

If $a \in \mathbb{T}$, $\sup \mathbb{T} = \infty$, and $f \in \mathrm{C_{rd}}$ on $[a, \infty)$, then we define the *improper integral* by

$$\int_a^\infty f(t)\Delta t := \lim_{b\to\infty} \int_a^b f(t)\Delta t$$

provided this limit exists, see Bohner and Guseinov [3].

The chain rule on $\mathbb{T}$ plays an important role in this paper (see the proof of Theorem 3.2) and is given as

$$(f \circ g)^\Delta(t) = \left\{ \int_0^1 f'\left(g(t) + h\mu(t)g^\Delta(t)\right) dh \right\} g^\Delta(t), \tag{7}$$

where $f : \mathbb{R} \mapsto \mathbb{R}$ is continuously differentiable and $g : \mathbb{T} \mapsto \mathbb{R}$ is delta differentiable, see Bohner and Peterson [4, Theorem 1.90].

In the earlier paper by Akın–Bohner [2], the asymptotic behavior of all positive decreasing solutions of (1) is considered. It is shown that any nontrivial solution of (1) is eventually monotone and belongs to one of the two classes:

$$M^+ := \{x \in S : \text{ there exists } T \geq t_0 \text{ such that } x(t)x^\Delta(t) > 0 \text{ for } t \geq T\}$$
$$M^- := \{x \in S : x(t)x^\Delta(t) < 0 \text{ on } [t_0, \infty)\},$$

where S is the set of nontrivial solutions of equation (1) on $[t_0, \infty)$, see [2, Lemma 3.1]. Concerning the class M^+ for equation (1), such a class can be empty when $\mathbb{T} = \mathbb{R}$, see Kiguradze and Chanturia [17, Corollary 17.4]. However, it is not true when $\mathbb{T} = \mathbb{Z}$. For instance,

$$x'' = x^2 \operatorname{sgn} x$$

does not have solutions in the class M^+, whereas the corresponding difference equation

$$\Delta^2 x_n = x_{n+1}^2 \operatorname{sgn} x_{n+1}$$

has positive increasing solutions.

We denote the subsets of M^+ consisting of bounded and unbounded solutions of (1) by M_B^+ and M_∞^+, respectively, where

$$M_B^+ = \{x \in M^+ : \lim_{t\to\infty} |x(t)| < \infty\}$$

and

$$M_\infty^+ = \{x \in M^+ : \lim_{t\to\infty} |x(t)| = \infty\}.$$

A solution $x \in M^+_\infty$ is said to be *strongly increasing* if $\lim_{t\to\infty} |a(t)\Phi_p(x^\Delta(t))| = \infty$ and *regularly increasing* otherwise. The set of strongly increasing solutions and the set of regularly increasing solutions will be denoted by $M^+_{\infty S}$ and $M^+_{\infty R}$, respectively, where

$$M^+_{\infty R} = \{x \in M^+ : \lim_{t\to\infty} |x(t)| = \infty, \lim_{t\to\infty} |a(t)\Phi_p(x^\Delta(t))| < \infty\}$$

and

$$M^+_{\infty S} = \{x \in M^+ : \lim_{t\to\infty} |x(t)| = \infty, \lim_{t\to\infty} |a(t)\Phi_p(x^\Delta(t))| = \infty\}.$$

Notice that

$$M^+ = M^+_B \cup M^+_\infty = M^+_B \cup M^+_{\infty R} \cup M^+_{\infty S}.$$

Such sets are characterized by certain integrals

$$\begin{aligned}
Y_1 &= \lim_{T\to\infty} \int_{t_0}^{T} \Phi_{p^*}\left(\frac{1}{a(t)}\right) \Phi_{p^*}\left(\int_{t_0}^{t} b(s)\Delta s\right) \Delta t,\\
Y_2 &= \lim_{T\to\infty} \int_{t_0}^{T} \Phi_{p^*}\left(\frac{1}{a(t)}\right) \Phi_{p^*}\left(\int_{t}^{T} b(s)\Delta s\right) \Delta t,\\
Y_3 &= \lim_{T\to\infty} \int_{t_0}^{T} \Phi_{p^*}\left(\frac{1}{a(t)}\right) \Delta t,\\
Y_4 &= \lim_{T\to\infty} \int_{t_0}^{T} b(t)\Delta t,
\end{aligned}$$

where Φ_{p^*} is the inverse of the map Φ_p, i.e., $\Phi_p(\Phi_{p^*}(u)) = \Phi_{p^*}(\Phi_p(u)) = u$. Then $\Phi_{p^*}(u) = |u|^{p^*-2}u$, where $\frac{1}{p} + \frac{1}{p^*} = 1$. One can find the proof of the following result in [2, Lemma 3.2] which gives the convergence or divergence of Y_1, Y_2, Y_3 and Y_4.

Lemma 2.1. *We have*

(i) If $Y_1 < \infty$, then $Y_3 < \infty$.

(ii) If $Y_2 < \infty$, then $Y_4 < \infty$.

(iii) If $Y_1 = \infty$, then $Y_3 = \infty$ or $Y_4 = \infty$.

(iv) If $Y_2 = \infty$, then $Y_3 = \infty$ or $Y_4 = \infty$.

(v) $Y_1 < \infty$ and $Y_2 < \infty$ if and only if $Y_3 < \infty$ and $Y_4 < \infty$.

3 Bounded Solutions of (1)

In this section the existence of bounded solutions of (1) is considered. We start with necessary and sufficient conditions ensuring that $M^+_B \neq \emptyset$.

Theorem 3.1. *Equation* (1) *has a solution in the class M^+_B if and only if $Y_1 < \infty$.*

Proof. Let $x \in M_B^+$. Without loss of generality assume $x(t) > 0$, $x^\Delta(t) > 0$ on $[t_0, \infty)$, $t \geq t_0$. Therefore, we have

$$\left[a(t)\Phi_p(x^\Delta)\right]^\Delta = b(t)f(x^\sigma) \geq b(t)m_f, \tag{8}$$

where $m_f = \min_{u\in[x(t_0),x(\infty)]} f(u)$. Integrating inequality (8) from t_0 to t yields

$$a(t)\Phi_p(x^\Delta(t)) \geq a(t_0)\Phi_p(x^\Delta(t_0)) + m_f \int_{t_0}^t b(s)\Delta s \geq m_f \int_{t_0}^t b(s)\Delta s$$

and so

$$x^\Delta(t) \geq \Phi_{p*}\left(\frac{m_f}{a(t)}\int_{t_0}^t b(s)\Delta s\right). \tag{9}$$

Integrating inequality (9) from t_0 to t yields

$$x(t) \geq x(t_0) + \int_{t_0}^t \Phi_{p^*}\left(\frac{m_f}{a(s)}\int_{t_0}^s b(\tau)\Delta\tau\right)\Delta s,$$

which is the desired result when $t \to \infty$.

Conversely, we prove that $M_B^+ \neq \emptyset$. Define $M_f = \max_{u\in[\frac{1}{2},1]} f(u)$ and choose $t_1 \geq t_0$ such that

$$\Phi_{p^*}(M_f)\left[\int_{t_1}^\infty \Phi_{p^*}\left(\frac{1}{a(t)}\int_{t_1}^t b(\tau)\Delta\tau\right)\right] \leq \frac{1}{2}.$$

Define X to be the Frechet space of all continuous functions on $[t_1, \infty)$ endowed with the topology of uniform convergence on compact subintervals of $[t_1, \infty)$. Let Ω be the nonempty subset of the Frechet space $C[t_1, \infty)$ given by

$$\Omega = \{u \in C[t_1, \infty) : \frac{1}{2} \leq u(t) \leq 1\}.$$

Clearly Ω is bounded, closed and convex. Now we consider the operator $T : \Omega \mapsto C[t_1, \infty)$ which assigns to any $u \in \Omega$ the continuous function $T(u) = y_u$ given by

$$y_u(t) = T(u)(t) = \frac{1}{2} + \int_{t_1}^t \Phi_{p^*}\left(\frac{1}{a(s)}\int_{t_1}^s b(\tau)f(u^\sigma(\tau))\Delta\tau\right)\Delta s.$$

Thus

$$\frac{1}{2} \leq T(u)(t) = \frac{1}{2} + \Phi_{p^*}(M_f)\left[\int_{t_1}^t \Phi_{p^*}\left(\frac{1}{a(s)}\int_{t_1}^s b(\tau)\Delta\tau\right)\Delta s\right] \leq 1,$$

which implies $T(\Omega) \subseteq \Omega$. One can show that T is continuous in $\Omega \subset X$ and relatively compact. Finally, using the Tychonov fixed point theorem gives the existence of a solution in M_B^+, similar to the argument in [2, Theorem 4.1]. □

The following result gives a stronger result.

Theorem 3.2. *Assume $Y_1 < \infty$ and*

$$\limsup_{|u|\to\infty} \frac{f(u)}{\Phi_p(u)} < \infty. \tag{10}$$

Then every solution x of equation (1) *in M^+ is bounded, i.e., $M_\infty^+ = \emptyset$.*

Proof. In view of (10) there exist two positive constants R and L such that

$$\frac{f(u)}{\Phi_p(u)} < L \quad \text{for} \quad u > R.$$

Assume there exists an unbounded solution x of equation (1) and without loss of generality assume $x(t) > R$, $x^\Delta(t) > 0$ for all $t \geq t_0$. From equation (1) we have

$$\begin{aligned}\left(\frac{a(t)\Phi_p(x^\Delta(t))}{\Phi_p(x(t))}\right)^\Delta &= \left(a(t)\Phi_p(x^\Delta(t))\right)^\Delta \frac{1}{\Phi_p(x^\sigma(t))} + a(t)\Phi_p(x^\Delta(t))\left(\frac{1}{\Phi_p(x(t))}\right)^\Delta \\ &= \frac{b(t)f(x^\sigma(t))}{\Phi_p(x^\sigma(t))} - \frac{a(t)\Phi_p(x^\Delta(t))[\Phi_p(x(t))]^\Delta}{\Phi_p(x(t))\Phi_p(x^\sigma(t))}.\end{aligned}$$

Integrating the above equation from t_0 to t gives

$$\frac{a(t)\Phi_p(x^\Delta(t)}{\Phi_p(x(t))} \leq H + L\int_{t_0}^t b(\tau)\Delta\tau, \tag{11}$$

where $H = \frac{a(t_0)\Phi_p(x^\Delta(t_0))}{\Phi_p(x(t_0))}$. If $Y_4 < \infty$, then there exists a positive constant H_1 such that

$$\frac{a(t)\Phi_p(x^\Delta(t))}{\Phi_p(x(t))} \leq H_1$$

or

$$\frac{x^\Delta(t)}{x(t)} \leq \Phi_{p^*}\left(\frac{H_1}{a(t)}\right).$$

Then by (7),

$$[\ln(x(t))]^\Delta = x^\Delta(t)\int_0^1 \frac{1}{x(t) + h\mu(t)x^\Delta(t)}dh \leq x^\Delta(t)\int_0^1 \frac{1}{x(t)}dh = \frac{x^\Delta(t)}{x(t)}.$$

This implies that

$$[\ln(x(t))]^\Delta \leq \frac{x^\Delta(t)}{x(t)} \leq \Phi_{p^*}\left(\frac{H_1}{a(t)}\right).$$

Integrating the above inequality from t_0 to t yields

$$\ln(x(t)) \leq \ln(x(t_0)) + \int_{t_0}^t \Phi_{p^*}\left(\frac{H_1}{a(\tau)}\right)\Delta\tau.$$

As $t \to \infty$, this contradicts the fact that $Y_1 < \infty$ implies $Y_3 < \infty$ by Lemma 2.1 (i).

If $Y_4 = \infty$, then choose $t_1 > t_0$ such that

$$H < L \int_{t_0}^{t_1} b(\tau)\Delta\tau.$$

From inequality (11) we have for $t \geq t_1$

$$\frac{a(t)\Phi_p(x^\Delta(t))}{\Phi_p(x(t))} \leq 2L \int_{t_0}^{t} b(\tau)\Delta\tau$$

or

$$\frac{x^\Delta(t)}{x(t)} \leq \Phi_{p^*}(2L)\Phi_{p^*}\left(\int_{t_0}^{t} \frac{b(\tau)}{a(t)}\Delta\tau\right).$$

By (7), we obtain

$$\ln(x(T)) \leq \ln(x(t_0)) + \Phi_{p^*}(2L)\int_{t_0}^{T} \Phi_{p^*}\left(\int_{t_0}^{t} \frac{b(s)}{a(t)}\Delta s\right)\Delta t.$$

As $T \to \infty$, this contradicts the fact that $Y_1 < \infty$. Therefore $M_\infty^+ = \emptyset$. □

Remark 3.1. In general Theorem 3.2 does not hold without the condition (10). When $\mathbb{T} = \mathbb{Z}$, Cecchi, Došlá and Marini in [7, Example 1] show that $x_n = (n-1)^{(4)}$ is an unbounded solution of the equation

$$\Delta((n-3)(n-4)\Delta x_n) = \frac{20}{n^{(4)}} f(x_{n+1}), \quad n > 4,$$

where $n^{(k)} = n(n-1)\cdots(n-k+1)$ and $f(u) = u^2 \operatorname{sgn} u$. In this case, $\dfrac{f(u)}{\Phi_p(u)} = u \operatorname{sgn} u$ and $Y_1 < \infty$.

The following corollary follows from Theorem 3.1, Lemma 1.1 and Theorem 3.2.

Corollary 3.1. *Assume $Y_1 < \infty$. Then $M_B^+ \neq \emptyset$ and $M_{\infty R}^+ = \emptyset$. In addition, if (10) holds, then $M_{\infty S}^+ = \emptyset$, i.e., every solution of (1) in M^+ is bounded.*

Proof. When $Y_1 < \infty$, $M_B^+ \neq \emptyset$ follows from Theorem 3.1. To prove $M_{\infty R}^+ = \emptyset$, we assume not. Then there exists a solution x of equation (1) in $M_{\infty R}^+$. Without loss of generality, we assume that $x(t) > 0$ and $x^\Delta(t) > 0$ for all $t \geq t_0$. Since $a(t)\Phi_p(x^\Delta(t))$ is bounded, there exists a positive constant m such that for all $t \geq t_0$,

$$x^\Delta(t) \leq \Phi_{p^*}\left(\frac{m}{a(t)}\right).$$

Integrating both sides of the above inequality from t_0 to t gives us

$$x(t) \leq x(t_0) + \int_{t_0}^{t} \Phi_{p^*}\left(\frac{m}{a(s)}\right)\Delta s.$$

Since $Y_1 < \infty$, $Y_3 < \infty$ by Lemma 2.1. But this contradicts the fact that x is a solution of equation (1) in $M_{\infty R}^+$ as $t \to \infty$. Finally, the last claim follows from Theorem 3.2. □

4 The Comparison Criterion

In this section, we consider two quasilinear dynamic equations

$$\left[a(t)\Phi_p(x^\Delta)\right]^\Delta = b(t)g(x^\sigma), \tag{12}$$

and

$$\left[a(t)\Phi_p(y^\Delta)\right]^\Delta = B(t)h(y^\sigma), \tag{13}$$

where a, b and B are real positive rd-continuous functions on $\mathbb{T}$ and functions $g, h : \mathbb{R} \mapsto \mathbb{R}$ are continuous with $ug(u) > 0$, $uh(u) > 0$ for $u \neq 0$ and $\Phi_p(u) = |u|^{p-2}u$ with $p > 1$.

The following comparison criterion gives the existence of upper solutions.

Theorem 4.1. *Suppose that $B(t) \geq b(t)$ and there exists a positive constant R such that*

$$|h(u)| \geq |g(u)| \ \text{ for } \ |u| \geq R \tag{14}$$

and h or g is strictly increasing for $|u| \geq R$. Let x be a solution of equation (12) such that $|x(t_0)| > R$, $x(t_0)x^\Delta(t_0) > 0$, $t_0 \in \mathbb{T}$. Then for any solution y of equation (13) in M^+ with $|y(t_0)| \geq |x(t_0)|$, $x(t_0)y(t_0) > 0$ and $\left|y^\Delta(t_0)\right| \geq \left|x^\Delta(t_0)\right|$ it holds for $t \geq t_0$

$$|y(t)| \geq |x(t)|$$

and

$$\left|a(t)\Phi_p(y^\Delta(t))\right| \geq \left|a(t)\Phi_p(x^\Delta(t))\right|.$$

Proof. Without loss of generality we consider solutions $x(t)$ starting with a positive value, i.e., $x(t_0) > 0$. In view of [2, Lemma 3.1], $x(t)$ and $y(t)$ are increasing and so $x(t) > R$, $y(t) > R$. Define for $t \geq t_0$

$$d(t) = y(t) - x(t).$$

Clearly, $d(t_0) \geq 0$ and $d^\Delta(t_0) \geq 0$. In order to finish the proof it is enough to show that d does not have a positive maximum in (t_0, ∞). Assume not, then there exists $t_1 \in (t_0, \infty)$ such that

$$d(t_1) = \max\{d(t) : t \in [t_0, \infty)\} > 0$$

and

$$d(t) < d(t_1) \text{ for } t > t_1.$$

One can show that $\rho(t_1) = t_1 < \sigma(t_1)$ is not possible and for cases $\rho(t_1) < t_1 < \sigma(t_1)$, $\rho(t_1) < t_1 = \sigma(t_1)$ and $\rho(t_1) = t_1 = \sigma(t_1)$ we obtain

$$d^\Delta(t_1) \leq 0 \text{ and } d^\Delta(\rho(t_1)) \geq 0.$$

Here, we refer to the paper by Akin [1] for the detail of the last claim. We define

$$G(t) = a(t)\left[\Phi_p(y^\Delta(t)) - \Phi_p(x^\Delta(t))\right].$$

Then we have

$$
\begin{aligned}
G^{\Delta}(t) &= B(t)h(y^{\sigma}(t)) - b(t)g(x^{\sigma}(t)) \\
&\geq b(t)\left[h(y^{\sigma}(t)) - g(x^{\sigma}(t))\right].
\end{aligned}
$$

By inequality (14) we obtain

$$
G^{\Delta}(t) \geq b(t)\left[h(y^{\sigma}(t)) - h(x^{\sigma}(t))\right], \quad G^{\Delta}(t) \geq b(t)\left[g(y^{\sigma}(t)) - g(x^{\sigma}(t))\right].
$$

Since $d(t_1) > 0$, the monotonicity of h or g gives $G^{\Delta}(\rho(t_1)) > 0$.

For the cases $\rho(t_1) < t_1 < \sigma(t_1)$ and $\rho(t_1) < t_1 = \sigma(t_1)$, we obtain $G(t_1) \leq 0$ and $G(\rho(t_1)) > 0$ by the monotonicity of Φ_p. On the other hand, by equation (6) we obtain

$$
G^{\sigma}(\rho(t_1)) = G(\rho(t_1)) + \mu(\rho(t_1))G^{\Delta}(\rho(t_1)) > 0.
$$

Since $\sigma(\rho(t_1)) = t_1$ in these cases, we have $G^{\sigma}(\rho(t_1)) = G(t_1)$. But this gives a contradiction.

For the case $\rho(t_1) = t_1 = \sigma(t_1)$, we obtain $G^{\Delta}(t_1) > 0$ and $G(t_1) = 0$ since $d(t_1) > 0$, $d^{\Delta}(t_1) = d^{\Delta}(\rho(t_1)) = 0$ and by the monotonicity of h or g. Therefore, there exists $\delta > 0$ such that $\lim_{t \to t_1^+} G^{\Delta}(t) = G^{\Delta}(t_1) > 0$. This implies that $G(t)$ is strictly increasing on $(t_1, t_1 + \delta]$ and so $G(t) > 0$ on $(t_1, t_1 + \delta]$. This ensures that $d^{\Delta}(t) > 0$ on $(t_1, t_1 + \delta]$, and so d is strictly increasing on $(t_1, t_1 + \delta]$. But this gives a contradiction.

Therefore, d does not have a positive maximum on (t_0, ∞) for all cases and so $d(t) \geq 0$ on $[t_0, \infty)$. Since $y^{\Delta}(t) - x^{\Delta}(t) = d^{\Delta}(t) \geq 0$ for $t \geq t_0$, the monotonicity of Φ_p yields the second part of the result. □

Remark 4.1. In the above result we have to assume that h or g is strictly increasing due to the case where $\rho(t_1) = t_1 = \sigma(t_1)$. Since we do not have dense points when $\mathbb{T} = \mathbb{Z}$, it is enough to assume that h or g is nondecreasing, see [7, Theorem 3]. When $\mathbb{T} = \mathbb{R}$, there exists a comparison result only for solutions of equations (12) and (13) when $h = K\Phi_p,\ K > 0$, see [8, Theorem 7].

5 Reciprocal Principle

In this section, we study the existence of unbounded solutions of equation (4) on $\mathbb{T}$. From now on we assume that $\mu(t)$ is delta-differentiable on $\mathbb{T}$. Let x be a solution of equation (4), then $y = a(t)\Phi_p(x^{\Delta}(t))$ is a solution of the reciprocal equation

$$
\left[\frac{1}{\Phi_{q^*}(b(t))}\Phi_{q^*}(y^{\Delta})\right]^{\Delta} = \left[1 + \mu^{\Delta}(t)\right]\frac{1}{\Phi_{p^*}(a^{\sigma}(t))}\Phi_{p^*}(y^{\sigma}), \tag{15}
$$

where p^* and q^* are conjugate numbers of p and q, respectively. Equation (15) follows from the equation (4) by replacing a with $\dfrac{1}{\Phi_{q^*}(b(t))}$ and b with $\left[1 + \mu^{\Delta}(t)\right]\dfrac{1}{\Phi_{p^*}(a^{\sigma}(t))}$, where we use $x^{\sigma\Delta} = \left[1 + \mu^{\Delta}\right] x^{\Delta\sigma}$, see Bohner and Tisdell [6]. Notice that for solutions x of equation (4) and y of equation (15) it holds that

$$x \in M^+ \text{ if and only if } y \in M^+.$$

Also, Y_3 for equation (4) plays the same role as Y_4 for equation (15) and vie versa, analogously Y_4 for equation (4) plays the same role as Y_3 for equation (15). Similarly, for equation (15) the integrals Y_1 and Y_2 become

$$Y_5 = \lim_{T\to\infty} \int_{t_0}^{T} b(t)\Phi_q \left(\int_{t_0}^{t} \left(1 + \mu^{\Delta}(s)\right) \Phi_{p^*} \left(\frac{1}{a^{\sigma}(s)} \right) \Delta s \right) \Delta t$$

and

$$Y_6 = \lim_{T\to\infty} \int_{t_0}^{T} b(t)\Phi_q \left(\int_{t}^{T} \left(1 + \mu^{\Delta}(s)\right) \Phi_{p^*} \left(\frac{1}{a^{\sigma}(s)} \right) \Delta s \right) \Delta t,$$

respectively. Because of $p \le q$ if and only if $q^* \le p^*$ and the reciprocity principle we have the following corollary.

Corollary 5.1. *We obtain*

(i) if $p \ge q$ and $Y_5 < \infty$, then every solution x of equation (4) *in M^+ satisfies $\lim_{t\to\infty} a(t)\Phi_p(x^{\Delta}(t)) < \infty$, i.e., $M^+_{\infty S} = \emptyset$;*

(ii) if $Y_5 = \infty$, then $a\Phi_p(x^{\Delta})$ of every solution x of equation (4) *in M^+ is unbounded;*

(iii) if $Y_1 = \infty$ and $Y_5 = \infty$, then every solution of equation (4) *in M^+ is strongly increasing, i.e., $M^+ = M^+_{\infty S} \neq \emptyset$;*

(iv) if $Y_1 = \infty$ and $Y_5 < \infty$, then equation (4) *has a solution in $M^+_{\infty R}$, i.e., $M^+_{\infty R} \neq \emptyset$.*

Proof. To prove part (i) we apply Theorem 3.2 to the reciprocal equation (15) and obtain that every solution of equation (15) in M^+ is bounded, i.e.,

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} a(t)\Phi_p(x^{\Delta}(t)) < \infty.$$

To prove part (ii) we apply Theorem 3.1 to the reciprocal equation (15). The proof of (iii) follows from Theorem 3.1 for equation (4) and part (ii). For part (iv) we apply Theorem 3.1 to obtain $M^+_B = \emptyset$ for equation (4), i.e., $M^+ = M^+_{\infty}$. Applying Theorem 3.1 to the reciprocal equation (15) completes the proof. □

6 Unbounded Solutions of (1)

Theorem 4.1 and Corollary 5.1 give us the existence of unbounded solutions of equation (1).

Theorem 6.1. *Assume* (10) *holds. If $Y_5 < \infty$, then equation* (1) *does not have solutions in $M^+_{\infty S}$, i.e., $M^+_{\infty S} = \emptyset$.*

Proof. Since (10) holds, there exist two positive constants L and R such that

$$f(u) \leq L\Phi_p(u) \text{ for } u \geq R.$$

Let $x(t)$ be a solution of equation (1) in $M^+_{\infty S}$ and without loss of generality assume $x(t) \geq R$ and $x^\Delta(t) > 0$ for $t \in [t_0, \infty)$. From Theorem 4.1 with $h(u) = L\Phi_p(u)$, $g(u) = f(u)$ and $B(t) = b(t)$ for any solution $y \in M^+$

$$\left[a(t)\Phi_p(y^\Delta(t))\right]^\Delta = b(t)L\Phi_p(y^\sigma(t)), \tag{16}$$

with $y(t_0) \geq x(t_0)$ and $y^\Delta(t_0) \geq x^\Delta(t_0)$ it holds for $t \geq t_0$

$$a(t)\Phi_p(y^\Delta(t)) \geq a(t)\Phi_p(x^\Delta(t)). \tag{17}$$

By Corollary 5.1 (i), equation (16) does not have solutions in $M^+_{\infty S}$. But inequality (17) gives a contradiction as $t \to \infty$. □

Remark 6.1. In general, Theorem 6.1 does not hold without the condition (10) when $\mathbb{T} = \mathbb{Z}$. In [7, Example 2] the equation

$$\Delta^2 x_n = \frac{2}{e^{n(n+1)} - 1} f(x_{n+1}), \quad n > 1,$$

is considered where $f(u) = |e^u - 1| \operatorname{sgn} u$. It is shown that $x_n = n(n-1)$ is an unbounded solution of the above equation and belongs to one of the class $M^+_{\infty s}$. In this case, for any $q > 1$, $Y_5 < \infty$ and (10) is not verified.

Theorem 6.2. *Assume that there exists $q > 1$ such that*

$$\limsup_{|u| \to \infty} \frac{f(u)}{\Phi_q(u)} < \infty. \tag{18}$$

If $Y_1 = \infty$ and $Y_5 < \infty$, then equation (1) *has a solution in $M^+_{\infty R}$, i.e., $M^+_{\infty R} \neq \emptyset$.*

Proof. By (18), there exist positive constants L and R such that

$$f(u) \leq L\Phi_q(u) \text{ for } u \geq R.$$

By Corollary 5.1(iv), there is a solution y of

$$\left[a(t)\Phi_p(y^\Delta(t))\right]^\Delta = b(t)L\Phi_q(y^\sigma(t))$$

in $M^+_{\infty R}$ and without loss of generality assume $y(t) \geq R$ and $y^\Delta(t) > 0$ for $t \in [t_0, \infty)$. Let x be a solution of equation (1) with $x(t_0) = y(t_0)$ and $x^\Delta(t_0) = y^\Delta(t_0)$. From Theorem 4.1 with $h(u) = L\Phi_q(u)$, $g(u) = f(u)$ and $B(t) = b(t)$, we obtain (17) and so $\limsup_{t\to\infty} a(t)\Phi_p(x^\Delta(t)) < \infty$ for $t \in [t_0, \infty)$. By Theorem 3.1, solution x belongs to M^+_∞ since $Y_1 = \infty$ and so $M^+_{\infty R} \neq \emptyset$. □

Remark 6.2. When $\mathbb{T} = \mathbb{Z}$, in [7, Example 3] the equation

$$\Delta\left[\Delta x_n\right] = e^{-n} f(x_{n+1}), \quad n \geq 1,$$

is considered where $f : \mathbb{R} \mapsto \mathbb{R}$ is continuous with $uf(u) > 0$ for $u \neq 0$ and $|f(u)| = e^{u^2}$ for $|u| \geq 1$ to show that the condition (18) cannot be dropped in Theorem 6.2.

We have the following corollary for equation (1). It follows from Theorems 3.1, 6.1 and 6.2.

Corollary 6.1. *Assume that $Y_1 = \infty$ and $Y_2 < \infty$. Then $M_B^+ = \emptyset$. In addition, if there exists $q > 1$ such that* (18) *holds, then $M_{\infty R}^+ \neq \emptyset$. If* (10) *holds, then $M_{\infty S}^+ = \emptyset$. Therefore, if there exists $q > 1$ such that* (18) *holds and* (10) *is verified, then every solution of* (1) *in M^+ is regularly increasing.*

Theorem 6.3. *Suppose that there exists $q > 1$ such that*

$$\liminf_{|u| \to \infty} \frac{f(u)}{\Phi_q(u)} > 0. \tag{19}$$

If $Y_1 = Y_5 = \infty$, then every solution of equation (1) *in M^+ belongs to $M_{\infty S}^+$, i.e., $M_{\infty R}^+ = M_B^+ = \emptyset$.*

Proof. By Theorem 3.1, it is enough to show that $M_{\infty R}^+ = \emptyset$ since $Y_1 = \infty$ implies $M_B^+ = \emptyset$. From (19), there exist positive constants l and R such that

$$f(u) \geq l\Phi_q(u) \text{ for } u \geq R.$$

Let x be a solution of equation (1) in $M_{\infty R}^+$ and without loss of generality assume $x(t) \geq R$ and $x^{\Delta}(t) > 0$ for $t \geq t_0$. Let z be a solution of equation

$$\left[a(t)\Phi_p(z^{\Delta}(t))\right]^{\Delta} = b(t) l \Phi_q(z^{\sigma}(t)$$

with $z(t_0) = x(t_0)$, $z^{\Delta}(t_0) = x^{\Delta}(t_0)$. From Theorem 4.1 with $h(u) = f(u)$, $g(u) = l\Phi_q(u)$ and $B(t) = b(t)$ it holds for $t \geq t_0$

$$a(t)\Phi(x^{\Delta}(t)) \geq a(t)\Phi_p(z^{\Delta}(t)). \tag{20}$$

By Corollary 5.1 (iii), z belongs to $M_{\infty S}^+$. But (20) gives a contradiction as $t \to \infty$. □

Remark 6.3. When $\mathbb{T} = \mathbb{Z}$, the condition (19) cannot be dropped in Theorem 6.3 for the equation

$$\Delta\left[\frac{1}{n+1}\Delta x_n\right] = \frac{2n}{n+2} f(x_{n+1}), \quad n > 1,$$

where $f : \mathbb{R} \mapsto \mathbb{R}$ is continuous with $uf(u) > 0$ for $u \neq 0$ and $|f(u)| = \frac{1}{|u|}$ for $|u| \geq 1$. It is shown that $x_n = n(n-1)$ belongs to the class $M_{\infty R}^+$. In this case, $Y_1 = \infty$, $Y_5 = \infty$ and $\lim_{u \to \infty} \frac{f(u)}{\Phi_q(u)} = 0$ for any $q > 1$, see [7, Example 4].

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