Existence of solutions for second-order dynamic inclusions

Elvan Akın-Bohner

Department of Mathematics and Statistics,
Missouri University of Science and Technology,
Rolla, MO 65409-0020, USA
E-mail: akine@mst.edu

Shurong Sun*

School of Science,
University of Jinan,
Jinan, Shandong 250022, China

and

Department of Mathematics and Statistics,
Missouri University of Science and Technology,
Rolla, MO 65409-0020, USA
E-mail: sshrong@163.com
*Corresponding author

Abstract: By means of the method of lower and upper solutions, we study the existence of solutions for the second-order dynamic inclusions on time scales. The presented studies extend some recent results both for dynamic inclusions and differential inclusions.

Keywords: dynamic inclusions; time scales; boundary value problem; existence of solutions.

Reference to this paper should be made as follows: Akın-Bohner, E. and Sun, S. (2011) 'Existence of solutions for second-order dynamic inclusions', *Int. J. Dynamical Systems and Differential Equations*, Vol. 3, Nos. 1/2, pp.24–37.

Biographical notes: Elvan Akın-Bohner is a Professor at Missouri University of Science and Technology. Her research interests include differential, difference and dynamic equations, oscillation theory, inequalities, boundary value problems, dynamical systems, time scales, Sturm-Liouville equations, functional dynamic equations, and hybrid systems.

Shurong Sun is a Professor at the School of Science, University of Jinan. Her research interests include oscillation of differential and difference equations, dynamic systems on time scales, Sturm-Liouville theory, spectral theory of Hamiltonian systems, boundary value problems, fractional differential equations, and hybrid systems.

1 Introduction

We consider the second-order dynamic inclusion

$$(p(t)y^{\Delta}(t))^{\Delta} \in F(t, y^{\sigma}(t)), \quad t \in [a, b]_{\mathbb{T}}^{\kappa} \tag{1.1}$$

associated with boundary conditions

$$y(a) = y(\sigma(b)) = 0, \tag{1.2}$$

where $\mathbb{T}$ is a time scale, and $a, b \in \mathbb{T}$ with $a < b$, $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$. We assume that $p \in \mathrm{C}_{\mathrm{rd}}([a, b]_{\mathbb{T}}^{\kappa}, \mathbb{R})$ and $F : [a, b]_{\mathbb{T}}^{\kappa} \times \mathbb{R} \to 2^{\mathbb{R}} \setminus \emptyset$ is a multifunction with compact and convex values such that $|F(t, u)| = \sup\{|y| : y \in F(t, u)\}$ and $F(t, u) > 0$ means $y > 0$ for each $y \in F(t, u)$. For a function $y \in \mathrm{C}([a, b]_{\mathbb{T}})$ we put $||y|| = \max\limits_{t \in [a,b]_{\mathbb{T}}} |y(t)|$.

By a solution y of (1.1)–(1.2), we mean there exists a function $u \in \mathrm{C}_{\mathrm{rd}}^{1}([a, b]_{\mathbb{T}}^{\kappa}, \mathbb{R})$ such that $u(t) \in F(t, y^{\sigma}(t))$ on $[a, b]_{\mathbb{T}}^{\kappa}$ satisfying $(p(t)y^{\Delta}(t))^{\Delta} = u(t)$ on $[a, b]_{\mathbb{T}}^{\kappa}$ and (1.2).

Assume that $g : [a, b]_{\mathbb{T}} \times [a, b]_{\mathbb{T}} \to \mathbb{R}$ is a single-valued function. By a solution y of integral inclusion

$$y(t) \in \int_{a}^{b} g(t, s)F(s, y^{\sigma}(s))\Delta s, \quad t \in [a, b]_{\mathbb{T}}, \tag{1.3}$$

we mean there exists a function $u \in \mathrm{C}_{\mathrm{rd}}^{1}([a, b]_{\mathbb{T}}^{\kappa}, \mathbb{R})$ such that $u(t) \in F(t, y^{\sigma}(t))$ on $[a, b]_{\mathbb{T}}^{\kappa}$ satisfying $y(t) = \int_{a}^{b} g(t, s)u(s)\Delta s$.

In Section 2, we briefly mention time scale calculus with preliminary results. An excellent introduction of time scale calculus is given by Bohner and Peterson (2001, 2003). In Section 3, we obtain existence of solutions of (1.1)–(1.2). For the last result in this section, we use the method of lower and upper solutions. This method on time scales first was developed by Akın-Bohner (2000). Our results generalise results by Bohner and Tisdell (2005). We refer the reader to manuscripts by Atıcı and Biles (2004) for first order dynamic inclusions on time scales and by Akın-Bohner and Sun (not dated) for second dynamic inclusions on time scales. In the last section, we highlight our main results with an example.

2 Time scale calculus and preliminary results

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. On any time scale we define the forward and backward jump operators by

$$\sigma(t) := \inf\{s \in \mathbb{T} \mid s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} \mid s < t\}.$$

A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$, right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. The graininess μ of the time scale is defined by $\mu(t) := \sigma(t) - t$. We put $\mathbb{T}^{k} = \mathbb{T}$ if $\mathbb{T}$ is unbounded above and otherwise $\mathbb{T}^{k} = \mathbb{T} \backslash (\rho(\sup\mathbb{T}), \sup\mathbb{T}]$.

For a function $f : \mathbb{T} \to \mathbb{R}$ the delta derivative $f^{\Delta}(t)$ at $t \in \mathbb{T}$ is defined to be the number (provided it exists) with the property such that for every $\varepsilon > 0$, there exists a neighbourhood U of t with

$$|f^{\sigma}(t) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \le \varepsilon|\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right-dense point and if there exists a finite left-sided limit at all left-dense points. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$.

f is said to be differentiable if its derivative exists. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous function is denoted by $\mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$.

The derivative and the shift operator σ are related by the formula

$$f^{\sigma} = f + \mu f^{\Delta}, \quad \text{where} \quad f^{\sigma} := f \circ \sigma.$$

For $a, b \in \mathbb{T}$ and a differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta}(t)\Delta t = f(b) - f(a).$$

A useful formula is

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t). \tag{2.1}$$

We will make use of the following derivatives of the product fg, the integrals $\int_a^t f(\tau)\Delta\tau$ and $\int_a^t f(t,\tau)\Delta\tau$.

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f^{\sigma}(t)g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g^{\sigma}(t), \tag{2.2}$$

$$\left(\int_a^t f(\tau)\Delta\tau\right)^{\Delta} = f(t), \tag{2.3}$$

$$\left(\int_a^t f(t,\tau)\Delta\tau\right)^{\Delta} = \int_a^t f^{\Delta}(t,\tau)\Delta\tau + f(\sigma(t), t), \tag{2.4}$$

where f^{Δ} denotes the delta derivative of f with respect to the variable t.

Definition 2.1: A function $\alpha \in \mathrm{C}([a,b]_{\mathbb{T}})$ is called a *lower solution* of (1.1) if

$$F(t, \alpha^{\sigma}(t)) \cap (-\infty, (p(t)\alpha^{\Delta}(t))^{\Delta}] \ne \emptyset, \quad t \in [a,b]^{\kappa}_{\mathbb{T}},$$
$$\alpha(a) \le 0, \quad \alpha(\sigma(b)) \le 0.$$

A function $\beta \in C([a,b]_{\mathbb{T}})$ is called an *upper solution* of (1.1) if

$$F(t, \beta^{\sigma}(t)) \cap [(p(t)\beta^{\Delta}(t))^{\Delta}, \infty) \ne \emptyset, \quad t \in [a,b]^{\kappa}_{\mathbb{T}},$$
$$\beta(a) \ge 0, \quad \beta(\sigma(b)) \ge 0.$$

In the next section, we obtain a solution staying between a lower solution and an upper solution of (1.1), and so we use the following result.

Lemma 2.1 (Bohner and Peterson, 2003, Lemma 6.2)**:** *Assume y, y^{Δ} are continuous on $[a,b]_{\mathbb{T}}$, py^{Δ} is delta differentiable on $[a,b]_{\mathbb{T}}^{\kappa}$ and $(py^{\Delta})^{\Delta}$ is rd-continuous on $[a,b]_{\mathbb{T}}^{\kappa^2}$. If there exists $c \in (a,b)_{\mathbb{T}}$ such that*

$$y(c) = \max\{y(t) : \ t \in [a,b]_{\mathbb{T}}\} \quad \textit{and} \quad y(t) < y(c) \quad \textit{for} \quad t \in (c,b]_{\mathbb{T}},$$

then

$$p^{\Delta}(c) \leq 0 \quad \textit{and} \quad (py^{\Delta})^{\Delta}(\rho(c)) \leq 0.$$

Remark 2.1: One can see that c in Lemma 2.1 cannot be right-scattered and left-dense so it implies that $\sigma(\rho(c)) = c$.

A map $G : \mathbb{R} \to \mathrm{CK}(\mathbb{R})$ ($\mathrm{CK}(\mathbb{R})$ denotes the set of nonempty, closed, and convex subsets of $\mathbb{R}$) is called *upper semicontinuous* provided $\{u_k\}_{k\in\mathbb{N}}$ and $\{v_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ with $u_k \to u$, $v_k \to v$ as $k \to \infty$ and $v_k \in G(u_k)$ for all $k \in \mathbb{N}$ always implies $v \in G(u)$.

Throughout this paper we assume that

(H1). $F : [a,b]_{\mathbb{T}} \times \mathbb{R} \to \mathrm{CK}(\mathbb{R})$ is such that $F(t,\cdot)$ is upper semicontinuous for all $t \in [a,b]_{\mathbb{T}}$.

Our main results are based on the following two existence principles extracted from Bohner and Tisdell (2005).

Lemma 2.2: *Assume (H1) and suppose that $g : [a,b]_{\mathbb{T}} \times [a,b]_{\mathbb{T}} \to \mathbb{R}$ is continuous. If for all $r > 0$ there exists a nonnegative function $h_r \in \mathrm{C}([a,b]_{\mathbb{T}})$ with*

$$|F(t,u)| \leq h_r(t) \quad \textit{for all} \quad t \in [a,b]_{\mathbb{T}} \quad \textit{and all} \quad |u| \leq r \tag{2.5}$$

and if there exists a constant M with $||y|| \neq M$ for all solutions y of integral inclusion

$$y(t) \in \lambda \int_a^b g(t,s)F(s,y^{\sigma}(s))\Delta s, \quad t \in [a,b]_{\mathbb{T}} \tag{2.6}$$

for all $\lambda \in (0,1)$, then

$$y(t) \in \int_a^b g(t,s)F(s,y^{\sigma}(s))\Delta s, \quad t \in [a,b]_{\mathbb{T}} \tag{2.7}$$

has a solution.

Lemma 2.3: *Assume (H1) and suppose that $g : [a,b]_{\mathbb{T}} \times [a,b]_{\mathbb{T}} \to \mathbb{R}$ is continuous. If there exists a nonnegative function $h \in \mathrm{C}([a,b]_{\mathbb{T}})$ with*

$$|F(t,u)| \leq h(t) \quad \textit{for all} \quad t \in [a,b]_{\mathbb{T}} \quad \textit{and all} \quad u \in \mathbb{R},$$

then (2.7) has a solution.

In Section 3, we will present an existence result for (1.1)–(1.2) subject to the assumption that the time scale $\mathbb{T}$ has a differentiable forward jump operator σ, and so we need the following results.

Lemma 2.4 (Bohner and Tisdell, 2005, Lemma 1)**:** *Assume σ is differentiable on $\mathbb{T}$. If $y : \mathbb{T} \to \mathbb{R}$ is differentiable, then so is $y^\sigma : \mathbb{T}^\kappa \to \mathbb{R}$, and we have*

$$y^{\sigma\Delta} = \sigma^\Delta y^{\Delta\sigma}. \tag{2.8}$$

Lemma 2.5: *Assume σ is differentiable on $\mathbb{T}$. If y^Δ and py^Δ are delta differentiable on $\mathbb{T}$, then so is $p(y^2)^\Delta$, and*

$$[p(t)(y^2)^\Delta(t)]^\Delta = 2y^\sigma(t)(p(t)y^\Delta(t))^\Delta + p(t)(y^\Delta(t))^2 + p^\sigma(t)\sigma^\Delta(y^{\Delta\sigma})^2, \quad t \in \mathbb{T}.$$

Proof: Firstly, we obtain

$$(y^2)^\Delta(t) = y(t)y^\Delta(t) + y^\sigma(t)y^\Delta(t)$$

by (2.2) and so

$$p(t)(y^2)^\Delta(t) = y(t)p(t)y^\Delta(t) + y^\sigma(t)p(t)y^\Delta(t).$$

Applying (2.8) and (2.2) again, we get

$$\begin{aligned}[p(t)(y^2)^\Delta(t)]^\Delta &= [y(t)(p(t)y^\Delta(t)) + y^\sigma(t)(p(t)y^\Delta(t))]^\Delta \\ &= y^\sigma(t)(p(t)y^\Delta(t))^\Delta + y^\Delta(t)p(t)y^\Delta(t) + y^\sigma(t)(p(t)y^\Delta(t))^\Delta \\ &\quad + y^{\sigma\Delta}(t)p^\sigma(t)y^{\Delta\sigma}(t) \\ &= 2y^\sigma(t)(p(t)y^\Delta(t))^\Delta + p(t)(y^\Delta(t))^2 + p^\sigma(t)\sigma^\Delta(y^{\Delta\sigma}(t))^2.\end{aligned}$$

□

In next section, we will apply the following fixed point theorem to study the existence of solutions to (1.1)–(1.2). We refer the reader to Agarwal et al. (2001, 2003).

Lemma 2.6: *Let E be a Banach space, U an open subset of E and $0 \in U$. Suppose $P : U \to \mathrm{CK}(E)$ is an upper semicontinuous and compact map. Then either P has a fixed point in $\bar{U}$ or there exists $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda P(u)$.*

To prove the compactness of the image of an upper semicontinuous map,we will use a criterion which can be found in Stehlík and Tisdell (2005).

Lemma 2.7: *Let X and Y be two normed spaces and $P : X \to Y$ be an upper semicontinuous map. If K is a compact set in X, then $P(K)$ is a compact set in Y.*

3 Existence of solutions of (1.1)–(1.2)

In this section, we investigate the existence of solutions for dynamic inclusion (1.1) with boundary condition (1.2). We first show the following equivalence.

Theorem 3.1: *Assume that* $(H1)$ *and the following condition are satisfied*

$$d = \int_a^{\sigma(b)} \frac{1}{p(t)} \Delta t < \infty \quad \textit{such that} \quad d \neq 0. \tag{3.1}$$

Define a function $g : [a,b]_{\mathbb{T}} \times [a,b]_{\mathbb{T}} \to \mathbb{R}$ *by*

$$g(t,s) := \begin{cases} -\frac{1}{d}\left(\int_a^{\sigma(s)} \frac{\Delta\tau}{p(\tau)}\right)\left(\int_t^{\sigma(b)} \frac{\Delta\tau}{p(\tau)}\right) & \textit{if } a \le \sigma(s) \le t \le \sigma(b) \\ -\frac{1}{d}\left(\int_a^{t} \frac{\Delta\tau}{p(\tau)}\right)\left(\int_{\sigma(s)}^{\sigma(b)} \frac{\Delta\tau}{p(\tau)}\right) & \textit{if } a \le t \le s \le \sigma(b) \end{cases}. \tag{3.2}$$

Then y *solves* (1.1)–(1.2) *if and only if* y *solves* (2.7).

Proof: First assume y solves (2.7). Then there exists a function $u \in F(t, y^{\sigma}(t))$ such that

$$\begin{aligned} y(t) &= \int_a^b g(t,s)u(s)\Delta s = \int_a^t g(t,s)u(s)\Delta s + \int_t^b g(t,s)u(s)\Delta s \\ &= -\frac{1}{d}\int_a^t \left[\int_a^{\sigma(s)} \frac{\Delta\tau}{p(\tau)} \int_t^{\sigma(b)} \frac{\Delta\tau}{p(\tau)}\right] u(s)\Delta s \\ &\quad -\frac{1}{d}\int_t^b \left[\int_a^{t} \frac{\Delta\tau}{p(\tau)} \int_{\sigma(s)}^{\sigma(b)} \frac{\Delta\tau}{p(\tau)}\right] u(s)\Delta s. \end{aligned}$$

Using (2.3) and (2.4), we get

$$\begin{aligned} y^{\Delta}(t) &= -\frac{1}{d}\int_a^t \left[-\frac{1}{p(t)}\int_a^{\sigma(s)} \frac{\Delta\tau}{p(\tau)} u(s)\right]\Delta s \\ &\quad -\frac{1}{d}\left(\int_a^{\sigma(t)} \frac{\Delta\tau}{p(\tau)}\right)\left(\int_{\sigma(t)}^{\sigma(b)} \frac{\Delta\tau}{p(\tau)}\right)u(t) \\ &\quad -\frac{1}{d}\int_t^b \left[\frac{1}{p(t)}\int_{\sigma(s)}^{\sigma(b)} \frac{\Delta\tau}{p(\tau)} u(s)\right]\Delta s \\ &\quad +\frac{1}{d}\left(\int_a^{\sigma(t)} \frac{\Delta\tau}{p(\tau)}\right)\left(\int_{\sigma(t)}^{\sigma(b)} \frac{\Delta\tau}{p(\tau)}\right)u(t) \\ &= \frac{1}{d\cdot p(t)}\int_a^t \left[\int_a^{\sigma(s)} \frac{\Delta\tau}{p(\tau)} u(s)\right]\Delta s - \frac{1}{d\cdot p(t)}\int_t^b \left[\int_{\sigma(s)}^{\sigma(b)} \frac{\Delta\tau}{p(\tau)} u(s)\right]\Delta s, \end{aligned}$$

and so

$$(p(t)y^{\Delta}(t))^{\Delta} = \frac{1}{d}\int_a^{\sigma(t)} \frac{\Delta\tau}{p(\tau)} u(t) + \frac{1}{d}\int_{\sigma(t)}^{\sigma(b)} \frac{\Delta\tau}{p(\tau)} u(t) = u(t).$$

Clearly, $y(a) = y(\sigma(b)) = 0$, and so y solves (1.1)–(1.2).

Conversely, assume that y solves (1.1)–(1.2). Then

$$y(t) = \int_a^b g(t,s)[p(s)y^{\Delta}(s)]^{\Delta}\Delta s, \tag{3.3}$$

see Bohner and Peterson (2001, Corollary 4.75), and

$$[p(t)y^{\Delta}(t)]^{\Delta} \in F(t, y^{\sigma}(t)), \quad t \in [a,b]_{\mathbb{T}}^{\kappa}.$$

This implies that

$$\int_a^b g(t,s)[p(s)y^{\Delta}(s)]^{\Delta}\Delta s \in \int_a^b g(t,s)F(s, y^{\sigma}(s))\Delta s,$$

which together with (1.3) and (3.3) show that y solves (2.7). This completes the proof. □

From Theorem 3.1, we can now give our first existence result for the second order dynamic inclusion (1.1) with boundary condition (1.2).

Theorem 3.2: *Assume that conditions* (*H1*) *and* (3.1) *hold, and*

(H2). *there exists a continuous and nondecreasing function* $\omega : [0,\infty) \to [0,\infty)$ *with* $\omega(u) > 0$ *for* $u > 0$ *and a function* $q : [a,b]_{\mathbb{T}} \to [0,\infty)$ *such that*

$$|F(t,u)| \le q(t)\omega(|u|) \quad \textit{for all} \quad u \in \mathbb{R} \quad \text{and} \quad t \in [a,b]_{\mathbb{T}}.$$

Define

$$Q_0 := \max_{t\in[a,b]_{\mathbb{T}}} \int_a^b |g(t,s)|q(s)\Delta s,$$

where g *is defined by* (3.2). *If*

$$\sup_{c>0} \frac{c}{\omega(c)} > Q_0, \tag{3.4}$$

then dynamic inclusion (1.1) *with boundary condition* (1.2) *has a solution.*

Proof: Suppose $M > 0$ satisfies

$$\frac{M}{\omega(M)} > Q_0. \tag{3.5}$$

Consider

$$(p(t)y^{\Delta}(t))^{\Delta} \in \lambda F(t, y^{\sigma}(t)), \quad t \in [a,b]_{\mathbb{T}},\ y(a) = y(\sigma(b)) = 0 \tag{3.6}$$

with $0 < \lambda < 1$. By Theorem 3.1, (3.6) is equivalent to (2.6). Let y be any solution of (2.6) for $0 < \lambda < 1$. By (H2), we obtain

$$|y(t)| \le \lambda \int_a^b |g(t,s)|q(s)\omega(|y^\sigma(s)|)\Delta s$$
$$\le \omega(||y||) \int_a^b |g(t,s)|q(s)\Delta s \le \omega(||y||)Q_0,$$

and so $\frac{||y||}{\omega(||y||)} \le Q_0$. If $||y|| = M$, then $\frac{M}{\omega(M)} \le Q_0$, which is a contradiction to (3.4). Hence the result follows from Lemma 2.2. □

Theorem 3.3: *Assume that conditions (H1) and* (3.1) *hold and there exist a nonnegative function* $h_r \in \mathrm{C}([a,b]_{\mathbb{T}})$ *with* (2.5) *for all* $r > 0$. *If there exist a lower solution* α *and an upper solution* β *of* (1.1) *with* $\alpha(t) \le \beta(t)$ *for all* $t \in [a,b]_{\mathbb{T}}$, *then* (1.1)–(1.2) *has a solution with* $\alpha(t) \le y(t) \le \beta(t)$ *for all* $t \in [a,b]_{\mathbb{T}}$.

Proof: Define

$$h(t,x) := \begin{cases} \alpha^\sigma(t) & \text{if } x < \alpha^\sigma(t) \\ \beta^\sigma(t) & \text{if } x > \beta^\sigma(t) \\ x & \text{otherwise,} \end{cases} \quad r(t,x) := \begin{cases} \varphi(x - \alpha^\sigma(t)) & \text{if } x < \alpha^\sigma(t) \\ \varphi(x - \beta^\sigma(t)) & \text{if } x > \beta^\sigma(t) \\ 0 & \text{otherwise,} \end{cases}$$

$$\varphi(x) := \begin{cases} x & \text{if } |x| \le 1 \\ \frac{x}{|x|} & \text{if } |x| > 1, \end{cases} \quad \Gamma_+(t,x) := \begin{cases} (-\infty, (p(t)\alpha^\Delta(t))^\Delta], & \text{if } x < \alpha^\sigma(t) \\ (p(t)\beta^\Delta(t))^\Delta, \infty), & \text{if } x > \beta^\sigma(t) \\ \mathbb{R}, & \text{otherwise.} \end{cases}$$

$$F_+(t,x) = F(t,h(t,x)) \cap \Gamma_+(t,x), \; F_+^*(t,x) = F_+(t,x) + r(t,x).$$

We apply Lemma 2.3 to the function $F_+^* : [a,b]_{\mathbb{T}} \times \mathbb{R} \to CK(\mathbb{R})$. Clearly, $\Gamma_+(t,\cdot)$ is upper semicontinuous for each $t \in [a,b]_{\mathbb{T}}$ and hence so is $F_+(t,\cdot)$ and therefore $F_+^*(t,\cdot)$. By Theorem 3.1, the problem of the modified dynamic inclusion

$$(p(t)y^\Delta(t))^\Delta \in F_+^*(t, y^\sigma(t)), \quad t \in [a,b]_{\mathbb{T}}^\kappa \tag{3.7}$$

with boundary condition

$$y(a) = y(\sigma(b)) = 0 \tag{3.8}$$

is equivalent to the problem

$$y(t) \in \int_a^b g(t,s)F_+^*(s, y^\sigma(s))\Delta s, \quad t \in [a,b]_{\mathbb{T}}^\kappa, \tag{3.9}$$

where g is given by (3.2). Since

$$|F_+^*(t,u)| \le |F_+(t,u)| + |r(t,u)| \le h_{||\beta||}(t) + 1,$$

Lemma 2.3 ensures that (3.9) has a solution $y \in \mathrm{C}([a,b]_{\mathbb{T}})$. Hence by the above equivalence of (3.7)–(3.8) and (3.9), we conclude that there exists a solution y of (3.7)–(3.8). It remains to show that

$$\alpha(t) \le y(t) \le \beta(t) \quad \text{for all} \quad t \in [a,b]_{\mathbb{T}}$$

holds. Assume there exists $m \in [a,b]_{\mathbb{T}}$ with $a < m < \sigma(b)$, $y(m) > \beta(m)$. Define $u := y - \beta$ and let $\theta \in [a,b]_{\mathbb{T}}$ be such that $\max\limits_{t\in[a,b]_{\mathbb{T}}} = u(\theta)$. Therefore $u(\theta) > 0$. By Lemma 2.1 we have that

$$(pu^{\Delta})(\rho(\theta)) \le 0. \tag{3.10}$$

On the other hand,

$$\begin{aligned}(py^{\Delta})^{\Delta}(\rho(\theta)) &\in F_{+}^{*}(\rho(\theta), y^{\sigma}(\rho(\theta))) = F_{+}^{*}(\rho(\theta), y(\theta)) \\ &= F_{+}(\rho(\theta), y(\theta)) + r(\rho(\theta), y(\theta)) = F_{+}(\rho(\theta), y(\theta)) + \varphi(y(\theta) - \beta(\theta)),\end{aligned}$$

where $\sigma(\rho(\theta)) = \theta$ and so there exists $\psi(\theta) \in [(p\beta^{\Delta})^{\Delta}(\rho(\theta)), \infty)$ such that

$$(py^{\Delta})^{\Delta}(\rho(\theta)) = \psi(\theta) + \varphi(y(\theta) - \beta(\theta)) \ge (p\beta^{\Delta})^{\Delta}(\rho(\theta)) + \varphi(y(\theta) - \beta(\theta)).$$

Thus,

$$(pu^{\Delta})^{\Delta}(\rho(\theta)) = (py^{\Delta})^{\Delta}(\rho(\theta)) - (p\beta^{\Delta})^{\Delta}(\rho(\theta)) \ge \varphi(y(\theta) - \beta(\theta)) > 0,$$

which is a contradiction to (3.10). Hence $y(t) \le \beta(t)$ for all $t \in [a,b]_{\mathbb{T}}$. Similarly, we can show that $\alpha(t) \le y(t)$ for all $t \in [a,b]_{\mathbb{T}}$. The proof is complete. □

The following theorem gives another approach for proving the uniqueness of solution of (1.1)–(1.2).

Theorem 3.4: *Assume that α and β are lower and upper solutions of* (1.1)*, respectively. If there exists a function $f : [t_0, \infty)_{\mathbb{T}} \to [0, \infty)$ with strictly increasing in u such that*

$$F(t,u) \le f(t,u) \quad \textit{for } (t,u) \in [a,\sigma(b)]_{\mathbb{T}} \times [0,\infty), \tag{3.11}$$

$$F(t,u) \ge f(t,u) \quad \textit{for } (t,u) \in [a,\sigma(b)]_{\mathbb{T}} \times (-\infty,0], \tag{3.12}$$

where $F(t,u) \le (\ge) f(t,u)$ means $y \le (\ge) f(t,u)$ for each $y \in F(t,u)$. Then

$$\alpha(t) \le \beta(t) \quad \textit{for all} \quad t \in [a,\sigma(b)]_{\mathbb{T}}.$$

Proof: Define $h := \alpha - \beta$. We claim that $h(t) \le 0$ on $[a,\sigma(b)]_{\mathbb{T}}$. Assume that this is not true. Then there exists $t_1 \in [a,\sigma(b)]_{\mathbb{T}}$ with $h(t_1) > 0$. Since $h(a) \le 0$ and $h(\sigma(b)) \le 0$, we can choose $t_0 \in (a,\sigma(b))_{\mathbb{T}}$ such that

$$h(t_0) = \max\{h(t) : \quad t \in [a,b]\} > 0$$

and

$$h(t) < h(t_0) \quad \text{for all} \quad t \in (t_0, \sigma(b)]_{\mathbb{T}}.$$

Then by Lemma 2.1

$$(ph^{\Delta})^{\Delta}(\rho(t_0)) \leq 0. \tag{3.13}$$

On the other hand, α and β are lower and upper solutions of (1.1), respectively, and so there exist φ and ψ such that

$$\varphi(t) \in F(t, \alpha^{\sigma}(t)) \cap (-\infty, (p(t)\alpha^{\Delta}(t))^{\Delta}]$$

and

$$\psi(t) \in F(t, \beta^{\sigma}(t)) \cap [(p(t)\beta^{\Delta}(t))^{\Delta}, \infty).$$

Thus from (3.11), (3.12) and note that $\sigma(\rho(t_0)) = t_0$ by Remark 2.1, we have

$$\begin{aligned}(ph^{\Delta})^{\Delta}(\rho(t_0)) &= (p\alpha^{\Delta})^{\Delta}(\rho(t_0)) - (p\beta^{\Delta})^{\Delta}(\rho(t_0)) \\ &\geq \varphi(\rho(t_0)) - \psi(\rho(t_0)) \geq f(t_0, \alpha^{\sigma}(\rho(t_0))) - f(t_0, \beta^{\sigma}(\rho(t_0))) \\ &= f(t_0, \alpha(t_0)) - f(t_0, \beta(t_0)) \\ &> 0,\end{aligned}$$

which is a contradiction to (3.13). □

The following results follows from Theorems 3.3 and 3.4.

Corollary 3.1: *Assume that conditions* (*H1*) *and* (3.1) *hold and* α *and* β *are lower and upper solutions of* (1.1), *respectively. If there exist a nonnegative function* $h_r \in \mathrm{C}([a,b]_{\mathbb{T}})$ *with* (2.5) *for all* $r > 0$ *and a function* $f : [t_0, \infty)_{\mathbb{T}} \to [0, \infty)$ *with strictly increasing in* u *such that* (3.11) *and* (3.12) *hold. Then* (1.1)–(1.2) *has a solution such that* $\alpha(t) \leq y(t) \leq \beta(t)$ *for all* $t \in [a,b]_{\mathbb{T}}$.

The following result follows from the fact that every solution of dynamic inclusion (1.1) with boundary condition (1.2) is also a lower and an upper solution.

Corollary 3.2: *Assume that the conditions on* F, α *and* β *hold as in Theorem* 3.4. *Then the solution of dynamic inclusion* (1.1) *with boundary condition* (1.2) *is unique.*

The following existence result hold only when a time scale has a differentiable forward jump operator σ.

Theorem 3.5: *Assume* (*H1*) *holds if there exist constants* $L, K \geq 0$ *with*

$$|F(t,u)| \leq \inf_{v \in F(t,u)} (Luv + K) \quad \textit{for all} \quad (t,u) \in [a,b]_{\mathbb{T}} \times \mathbb{R}, \tag{3.14}$$

then any solution y *of* (1.1)–(1.2) *satisfies*

$$||y|| \leq K \max_{t\in[a,b]_{\mathbb{T}}} \int_a^b |g(t,s)|\Delta s,$$

where g *is defined by* (3.2) *and* σ *is differentiable on* $\mathbb{T}$.

Proof: Suppose y solves (1.1)–(1.2). Then, by Theorem 3.1, y also solves (2.7), i.e., there exists $w(t) \in F(t, y^\sigma(t)),\ t \in [a,b]_{\mathbb{T}}$ such that

$$y(t) = \int_a^b g(t,s)w(s)\Delta s \quad \text{for all} \quad t \in [a,b]_{\mathbb{T}}.$$

Since $w(t) \in F(t, y^\sigma(t))$, noting that σ is an increasing function and by Lemma 2.5, we have

$$\begin{aligned} |w(t)| \leq \inf_{v(t)\in F(t,y^\sigma(t))} (Ly^\sigma(t)v(t) + K) &\leq Ly^\sigma(t)(p(t)y^\Delta(t))^\Delta + K \\ &\leq \frac{L}{2}[p(t)(y^2)^\Delta(t)]^\Delta + K. \end{aligned}$$

Thus,

$$|y(t)| \leq \int_a^b |g(t,s||w(s)|\Delta s \leq \int_a^b |g(t,s| \left\{\frac{L}{2}[p(s)(y^2)^\Delta(s)]^\Delta + K\right\}\Delta s. \quad (3.15)$$

From (3.2), (2.1) and (2.2), we have

$$\begin{aligned} &\int_a^b |g(t,s|[p(s)(y^2)^\Delta(s)]^\Delta \Delta s \\ &= \frac{1}{d}\int_a^t \left(\int_a^{\sigma(s)} \frac{\Delta\tau}{p(\tau)}\right)\left(\int_t^{\sigma(b)} \frac{\Delta\tau}{p(\tau)}\right)[p(s)(y^2)^\Delta(s)]^\Delta \Delta s \\ &\quad + \frac{1}{d}\int_t^b \left(\int_a^t \frac{\Delta\tau}{p(\tau)}\right)\left(\int_{\sigma(s)}^{\sigma(b)} \frac{\Delta\tau}{p(\tau)}\right)[p(s)(y^2)^\Delta(s)]^\Delta \Delta s \\ &= \frac{1}{d}\left(\int_t^{\sigma(b)} \frac{\Delta\tau}{p(\tau)}\right)\int_a^t \left\{\left[\int_a^s \frac{\Delta\tau}{p(\tau)} p(s)(y^2)^\Delta(s)\right]^\Delta \right. \\ &\quad \left. - \left(\int_a^s \frac{\Delta\tau}{p(\tau)}\right)^\Delta p(s)(y^2)^\Delta(s)\right\}\Delta s \\ &\quad + \frac{1}{d}\left(\int_a^t \frac{\Delta\tau}{p(\tau)}\right)\int_t^b \left\{\left[\int_s^{\sigma(b)} \frac{\Delta\tau}{p(\tau)} p(s)(y^2)^\Delta(s)\right]^\Delta \right. \\ &\quad \left. - \left(\int_s^{\sigma(b)} \frac{\Delta\tau}{p(\tau)}\right)^\Delta p(s)(y^2)^\Delta(s)\right\}\Delta s \end{aligned}$$

$$
\begin{aligned}
&= -y^2(t) - \frac{1}{d}\left(\int_a^t \frac{\Delta\tau}{p(\tau)}\right)\left(\int_b^{\sigma(b)} \frac{\Delta\tau}{p(\tau)} p(b)(y^2)^{\Delta}(b) + y^2(b)\right) \\
&= -y^2(t) - \frac{1}{d}\left(\int_a^t \frac{\Delta\tau}{p(\tau)}\right)\left(\mu(b)\frac{1}{p(b)} p(b)(y^2)^{\Delta}(b) + y^2(b)\right) \\
&= -y^2(t) - \frac{1}{d}(y^2)^{\sigma}(b)\left(\int_a^t \frac{\Delta\tau}{p(\tau)}\right) = -y^2(t). \qquad (3.16)
\end{aligned}
$$

Using (3.16) in (3.15), we obtain

$$|y(t)| \le -y^2(t) + K\int_a^b |g(t,s)|\Delta s \le K\int_a^b |g(t,s)|\Delta s,$$

which completes the proof. □

Define R by

$$R := K \max_{t\in[a,b]_{\mathbb{T}}} \int_a^b |g(t,s)|\Delta s,$$

where g is defined by (3.2).

Theorem 3.6: *Assume (H1) and* (3.14) *hold. If σ is differentiable on $[a,b]_{\mathbb{T}}$, then* (1.1)–(1.2) *has a solution.*

Proof: Multiplying both sides of the inequality in (3.14) by $\lambda \in [0,1]$, we find

$$|\lambda F(t,u)| \le \inf_{v\in\lambda F(t,u)} (Luv + K) \quad \text{for all} \quad (t,u) \in [a,b]_{\mathbb{T}} \times \mathbb{R}.$$

Therefore, by Theorem 3.5

$$(p(t)y^{\Delta}(t))^{\Delta} \in \lambda F(t, y^{\sigma}(t)), \quad t \in [a,b]_{\mathbb{T}},\ y(a) = y(\sigma(b)) = 0 \qquad (3.17)$$

has solutions such that $||y|| \le R$. Define the operators

$$\mathcal{F} : \mathrm{C}([a,b]_{\mathbb{T}}) \to \mathrm{CK}(\mathrm{C}([a,b]_{\mathbb{T}}))$$

by

$$\mathcal{F}(u) := \{v \in \mathrm{C}([a,b]_{\mathbb{T}}) : v(t) \in F(t, u^{\sigma}(t)) \quad \text{for all} \quad t \in [a,b]_{\mathbb{T}}\}$$

and

$$A : \mathrm{C}([a,b]_{\mathbb{T}}) \to \mathrm{C}([a,b]_{\mathbb{T}})$$

by

$$(Ay)(t) := \int_a^b g(t,s)y(s)\Delta s, \quad t \in [a,b]_{\mathbb{T}},$$

where g is defined by (3.2). It is clear that A is a linear and continuous operator. From Theorem 3.1, (3.17) is equivalent to the fixed point problem

$$y \in \lambda(A \circ \mathcal{F})(y).$$

Choose U to be the set

$$U := \{y \in \mathrm{C}([a,b]_{\mathbb{T}}) : \ ||y|| \leq R+1\}.$$

Now we will apply Lemma 2.6 to the map $A \circ \mathcal{F}$ by showing that

$$A \circ \mathcal{F} : \overline{U} \to \mathrm{CK}(\mathrm{C}([a,b]_{\mathbb{T}})) \text{ is upper semicontinuous and compact.}$$

Let $\{u_k\}_{k\in\mathbb{N}}$ and $\{w_k\}_{k\in\mathbb{N}}$ such that $u_k \to u_0, w_k \to w_0$ as $k \to \infty$, and $w_k \in (A \circ \mathcal{F})(u_k)$ for all $k \in \mathbb{N}$. Then there exists $v_k \in \mathcal{F}(u_k)$ with $w_k = Av_k$. Since F is upper semicontinuous and $\overline{U}$ is compact set, we can deduce from Lemma 2.7 that $\mathcal{F}(\overline{U})$ is a compact set. This implies that there exists at least a subsequence $\{v_{k_i}\}_{i\in\mathbb{N}}$ of $\{v_k\}_{k\in\mathbb{N}}$, say $v_{k_i} \to v_0$ as $i \to \infty$. Now $v_{k_i} \to v_0$ and $u_{k_i} \to u_0$ as $i \to \infty$ and $v_{k_i}(t) \in F(t, u^{\sigma}_{k_i}(t))$ for all $t \in [a,b]_{\mathbb{T}}$. Thus, since $F(t,\cdot)$ is upper semicontinuous for all $t \in [a,b]_{\mathbb{T}}$, we may conclude $v_0(t) \in F(t, u_0^{\sigma}(t))$ for all $t \in [a,b]_{\mathbb{T}}$ and therefore $v_0 \in \mathcal{F}(u_0)$. Since $v_{k_i} \to v_0$ as $i \to \infty$ and $A : \mathrm{CK}([a,b]_{\mathbb{T}}) \to \mathrm{CK}([a,b]_{\mathbb{T}})$ is continuous, we see that $w_{k_i} = Av_{k_i} \to Av_0$ as $i \to \infty$, and hence $w_0 = Av_0 \in (A \circ \mathcal{F})(u_0)$. Therefore $A \circ \mathcal{F} : \overline{U} \to \mathrm{CK}(\mathrm{C}([a,b]_{\mathbb{T}}))$ is upper semicontinuous we can use Lemma 2.7 to obtain the compactness of $A \circ \mathcal{F}$. Now we are ready to use Lemma 2.6, and thanks to the conclusion of Theorem 3.5, we can exclude the second possibility in Lemma 2.6. Therefore the operator $A \circ \mathcal{F}$ has a fixed point and the problem (1.1)–(1.2) has a solution. □

4 An example

Let us consider the following dynamic inclusion

$$y^{\Delta\Delta} \in F(t, y^{\sigma}(t)), \quad t \in [a,b]^{\kappa}_{\mathbb{T}} \tag{4.1}$$

with the boundary condition

$$y(a) = y(\sigma(b)) = 0, \tag{4.2}$$

where $F : [a,b]_{\mathbb{T}} \times \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ is a multivalued map defined by

$$(t,u) \to F(t,u) := [\cos u - 2, \cos u].$$

One can see easily that (3.1) holds, and F satisfies (H1), (H2), and (3.4). Hence (4.1)–(4.2) has a solution by Theorem 3.2.

Furthermore, F also satisfies (2.5). First note that $\alpha(t) = 0$ is a lower solution on $[a,b]_{\mathbb{T}}$ since

$$\alpha^{\Delta\Delta}(t) = 0 \in [-1,1] = [\cos \alpha^{\sigma}(t) - 2, \cos \alpha^{\sigma}(t)] = F(t, \alpha^{\sigma}(t))$$

and

$$\alpha(a) = 0 \leq 0, \ \alpha(\sigma(b)) = 0 \leq 0.$$

Next, let

$$\beta(t) = \int_a^t (c - s)\Delta s \quad \text{where } c = \frac{1}{\sigma(b) - a} \int_a^{\sigma(b)} s\Delta s.$$

Then

$$\beta^{\Delta\Delta}(t) = -1 \in [\cos\beta^\sigma(t) - 2, \cos\beta^\sigma(t)] = F(t, \beta^\sigma(t))$$

and

$$\beta(a) = 0 \geq 0, \quad \beta(\sigma(b)) = \int_a^{\sigma(b)} (c - s)\Delta s = 0 \geq 0,$$

so $\beta(t)$ is an upper solution on $[a, b]_{\mathbb{T}}$. Therefore we can conclude that there is a solution $y(t)$ satisfying $0 \leq y(t) \leq \int_a^t (c - s)\Delta s$ on $[a, b]_{\mathbb{T}}$ by Theorem 3.3.

Acknowledgements

This research is supported by the Natural Science Foundation of China (60774004), Shandong Research Funds (Y2008A28), and the University of Jinan Research Funds for Doctors (B0621).

References

Agarwal, R.P., Meehan, M. and O'Regan, D. (2001) *Fixed Point Theory and Applications*, Cambridge University Press, Cambridge.

Agarwal, R.P., O'Regan, D. and Lakshmikantham, V. (2003) 'Discrete second order inclusions', *J. Difference Equ. Appl.*, Vol. 9, pp.879–885.

Akın-Bohner, E. (2000) 'Boundary value problems for a differential equation on a measure chain', *Panamer. Math. J.*, Vol. 10, pp.17–30.

Akın-Bohner, E. and Sun, S. (not dated) *Oscillation Criteria for a Class of Second-Order Dynamic Inclusions*, Submitted.

Atıcı, F.M. and Biles, D.C. (2004) 'First order dynamic inclusions on time scales', *J. Math. Anal. Appl.*, Vol. 292, pp.222–237.

Bohner, M. and Tisdell, C.C. (2005) 'Second order dynamic inclusions', *Journal of Nonlinear Mathematical Physics*, Vol. 12, pp.36–45.

Bohner, M. and Peterson, A. (2001) *Dynamic Equations on Time Scales, An Introduction with Applications*, Birkhäuser, Boston.

Bohner, M. and Peterson, A. (2003) *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston.

Stehlík, P. and Tisdell, C.C. (2005) 'On boundary value problems for second-order discrete inclusions', *Boundary Value Problems*, Vol. 2005, pp.153–163.