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GRONWALL-LIKE INEQUALITIES ON TIME SCALES WITH APPLICATIONS

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Abstract

Some new nonlinear dynamic integral inequalities of Gronwall type for retarded functions are established. These inequalities can be used as basic tools in the study of certain classes of functional dynamic equations as well as dynamic delay equations.

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1 Introduction

Motivated by Agarwal et al. papers [3, 4], our purpose is to obtain time scales versions of some Gronwall-like inequalities used in the theory of differential and integral equations. It is well known that Gronwall–like inequalities in continuous and discrete cases play a crucial rule in studying the qualitative behavior of differential and difference equations. These inequalities have been used to investigate the global existence, uniqueness, boundedness and other properties of solutions of various nonlinear differential and difference equations. For the background and the summary on these particular subjects, we refer the interested reader to the excellent monographs [9]-[11] by Pachpatte and [1] by Agarwal. Many authors

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have studied some fundamental inequalities used in analysis on time scales, for example, see [2], [5, 6], [13, 14].

In this paper, we would like to study certain classes of functional dynamic equations as well as dynamic delay equations, see Section 3. In Section 1, we give a brief introduction to calculus on time scales as well as some important references. In Section 2, we obtain certain type of inequalities that are important to prove the main results in this paper.

For completeness, a few hints concerning the background of time scales, which has recently received a lot of attention, might be in order. In 1988, Stefan Hilger [12] in his Ph.D. thesis added a new wrinkle to the calculus by introducing the calculus on a time scale, which is a unification and extension of the theories of continuous and discrete analysis. A time scale is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$, and we usually denote it by the symbol $\mathbb{T}$. The two most popular examples are $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}$. Some other interesting time scales exist, and they give rise to plenty of applications such as the study of population dynamics models (see [7], pages 15 and 71). We define the forward and backward jump operators σ, $\rho:\mathbb{T}\to\mathbb{T}$ by

$$\sigma(t):=\inf\{s\in\mathbb{T}:s>t\} \text{ and } \rho(t):=\sup\{s\in\mathbb{T}:s<t\}$$

(supplemented by $\inf\emptyset=\sup\mathbb{T}$ and $\sup\emptyset=\inf\mathbb{T}$). A point $t\in\mathbb{T}$ with $t>\inf\mathbb{T}$ is called *right-scattered, right-dense, left-scattered* and *left-dense* if $\sigma(t)>t$, $\sigma(t)=t$, $\rho(t)<t$ and $\rho(t)=t$ holds, respectively. Points that are left-dense and right-dense at the same time are called *dense*. The set $\mathbb{T}^\kappa$ is derived from $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa=\mathbb{T}-\{m\}$. Otherwise, $\mathbb{T}^\kappa=\mathbb{T}$. *The graininess function* $\mu:\mathbb{T}\to[0,\infty)$ is defined by

$$\mu(t):=\sigma(t)-t.$$

Hence the graininess function is 0 if $\mathbb{T}=\mathbb{R}$ while it is 1 if $\mathbb{T}=\mathbb{Z}$. Let f be a function defined on $\mathbb{T}$, then we define the delta derivative of f at $t\in\mathbb{T}^\kappa$, denoted by $f^\Delta(t)$, to be the number (provided it exists) with the property such that for every $\varepsilon>0$, there exists a neighborhood $\mathbb{U}$ of t with

$$\left|f(\sigma(t))-f(s)-f^\Delta(t)\,[\sigma(t)-s]\right|\le\varepsilon\,|\sigma(t)-s| \qquad \text{for all } s\in\mathbb{U}.$$

Some elementary facts concerning the delta derivative are:

- If f is differentiable at t, then

$$f^\sigma(t)=f(\sigma(t))=f(t)+\mu(t)f^\Delta(t).$$

- If f and g are differentiable at t, then fg is differentiable at t with

$$(fg)^\Delta(t)=f^\sigma(t)g^\Delta(t)+f^\Delta(t)g(t)=f(t)g^\Delta(t)+f^\Delta(t)g^\sigma(t).$$

- If f and g are differentiable at t and $g(t)g(\sigma(t))\neq0$, then $\frac{f}{g}$ is differentiable at t with

$$\left(\frac{f}{g}\right)^\Delta(t)=\frac{f^\Delta(t)g(t)-f(t)g^\Delta(t)}{g(t)g(\sigma(t))}.$$

We say $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous $(f \in \mathrm{C_{rd}}(\mathbb{T},\mathbb{R}))$ provided f is continuous at right-dense points in $\mathbb{T}$ and its left-sided limit exists (finite) at left-dense points in $\mathbb{T}$. The importance of rd-continuous functions is that *every rd-continuous function possesses an antiderivative*. A function $F : \mathbb{T}^\kappa \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$. In this case we define the integral of f by

$$\int_a^t f(s)\Delta s = F(t) - F(a)$$

for all $t \in \mathbb{T}$. Other useful formulas are as follows:

$$\int_t^{\sigma(t)} f(s)\Delta s = \mu(t)f(t),$$

$$\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t,$$

$$\int_a^b f(t)g^\Delta(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^\Delta(t)g^\sigma(t)\Delta t.$$

Let f, $g : \mathbb{T} \to \mathbb{R}$ be rd-continuous and a, $b \in \mathbb{T}$. If $|f(t)| \le g(t)$ on $[a,b)$, then

$$\left|\int_a^b f(t)\Delta t\right| \le \int_a^b g(t)\Delta t.$$

The following result is a chain rule on $\mathbb{T}$, see [7, Theorem 1.90].

Lemma 1.1. *Let $f : \mathbb{R} \mapsto \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \mapsto \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \mapsto \mathbb{R}$ is delta differentiable and the formula*

$$(f \circ g)^\Delta(t) = \left\{\int_0^1 f'\left(g(t) + h\mu(t)g^\Delta(t)\right) dh\right\} g^\Delta(t)$$

holds.

A comprehensive and an excellent treatment of calculus on time scales can be found, for instance, in [7, 8].

For convenience of notation, we let throughout

$$t_0 \in \mathbb{T}, \qquad \mathbb{T}_{t_0} = [t_0,\infty) \cap \mathbb{T},$$

and

$$a,b \in \mathbb{T} \qquad \mathbb{T}_{[a,b]} = [a,b] \cap \mathbb{T}.$$

We let $\mathbb{R}_0 = [0,\infty)$, $\mathbb{R}_1 = [1,\infty)$, and $C^1(M,N)$ be the class of all continuously differentiable functions defined on the set M to the set N.

2 Gronwall-Like Inequalities

In this section, we consider Gronwall type inequalities which will be useful to obtain the global existence of solutions for certain delay dynamic equations.

Theorem 2.1. *Let b, f_i, $g_i \in \mathrm{C_{rd}}(\mathbb{T}_{t_0}, \mathbb{R}_0)$, $i = 1, 2, \ldots, l$ such that b is nondecreasing and let $\alpha : \mathbb{T}_{t_0} \mapsto \mathbb{T}$ be nondecreasing such that $\alpha(t) \le t$ and $-\infty < a = \inf\{\alpha(s) : s \in \mathbb{T}_{t_0}\}$. Suppose that $q \ge 0$ is a constant, $\varphi \in C^1(\mathbb{R}_0, \mathbb{R}_0)$ is an increasing function with $\varphi(\infty) = \infty$ on $\mathbb{R}_0$, and ψ is a nondecreasing continuous function for $u \in \mathbb{R}_0$ with $\psi(u) > 0$ for $u > 0$. If $u : \mathbb{T}_a \mapsto \mathbb{R}_0$ and*

$$\varphi(u(t)) \le b(t) + \sum_{i=1}^{l} \int_{t_0}^{t} u^q(\alpha(s)) \left[f_i(s)\psi(u(\alpha(s))) + g_i(s)\right] \Delta s$$

for $t \in \mathbb{T}_{t_0}$, then

$$u(t) \le \varphi^{-1}\left\{G^{-1}\left[\Omega^{-1}\left(\Omega\left(G(b(t) + \int_{t_0}^{t} \sum_{i=1}^{l} g_i(s)\Delta s\right) + \int_{t_0}^{t} \sum_{i=1}^{l} f_i(s)\Delta s\right)\right]\right\} \tag{2.1}$$

for $t \in \mathbb{T}_{[t_0,\beta]}$, where

$$G(r) = \int_{r_0}^{r} \frac{ds}{(\varphi^{-1}(s))^q}, \quad r \ge r_0 > 0,$$

$$\Omega(r) = \int_{r_0}^{r} \frac{ds}{\psi[\varphi^{-1}(G^{-1}(s))]}, \quad r \ge r_0 > 0,$$

G^{-1}, and Ω^{-1} denote the inverse functions of G, Ω, respectively and $\beta \ge t_0$ is chosen such that

$$\Omega\left(G(b(t)) + \int_{t_0}^{t} \sum_{i=1}^{l} g_i(s)\Delta s\right) + \int_{t_0}^{t} \sum_{i=1}^{l} f_i(s)\Delta s \in Dom\left(\Omega^{-1}\right)$$

holds.

Proof. Let $\varepsilon > 0$ and $T \in \mathbb{T}_{t_0}$. Define a function $z : \mathbb{T}_{[t_0,T]} \to \mathbb{R}_0$ by

$$z(t) = \varepsilon + b(T) + \sum_{i=1}^{l} \int_{t_0}^{t} u^q(\alpha(s)) \left[f_i(s)\psi(u(\alpha(s))) + g_i(s)\right] \Delta s. \tag{2.2}$$

Clearly, $z(t)$ is nondecreasing, $u(t) \le \varphi^{-1}(z(t))$ for $t \in \mathbb{T}_{[t_0,T]}$ and $z(t_0) = \varepsilon + b(T)$. From (2.2), we obtain

$$\begin{aligned} z^{\Delta}(t) &= \sum_{i=1}^{l} u^q(\alpha(t)) \left[f_i(t)\psi(u(\alpha(t))) + g_i(t)\right] \\ &\le \left[\varphi^{-1}(z(t))\right]^q \sum_{i=1}^{l} \left[f_i(t)\psi(\varphi^{-1}(z(t))) + g_i(t)\right], \end{aligned} \tag{2.3}$$

for $t \in \mathbb{T}^{\kappa}_{[t_0,T]}$. Here we use the fact that $\alpha(t) \le t$ yields $z(\alpha(t)) \le z(t)$, φ is increasing, and $q > 0$ implies that

$$u^q(\alpha(t)) \le \left[\varphi^{-1}(z(\alpha(t)))\right]^q \le \left[\varphi^{-1}(z(t))\right]^q, \quad t \in \mathbb{T}_{[t_0,T]}.$$

Using the monotonicity of φ^{-1} and z, we obtain

$$\left[\varphi^{-1}(z(t))\right]^q \ge \left[\varphi^{-1}(z(0))\right]^q = \left[\varphi^{-1}(\varepsilon + b(T))\right]^q > 0, \quad t \in \mathbb{T}_{[t_0,T]}.$$

Hence it follows from (2.3) we have that

$$\frac{z^{\Delta}(t)}{[\varphi^{-1}(z(t))]^q} \le \sum_{i=1}^{l} \left[f_i(t)\psi(\varphi^{-1}(z(t))) + g_i(t)\right], \quad t \in \mathbb{T}^{\kappa}_{[t_0,T]}. \tag{2.4}$$

On the other hand, taking into account Lemma 1.1 and the definition of G, we have

$$\begin{aligned} (G(z(t)))^{\Delta} &= (G \circ z)^{\Delta}(t) \\ &= z^{\Delta}(t) \int_0^1 G'\left(z(t) + \mu(t) h z^{\Delta}(t)\right) dh, \end{aligned} \tag{2.5}$$

for $t \in \mathbb{T}^{\kappa}_{[t_0,T]}$, where $G'(t) = \frac{1}{[\varphi^{-1}(t)]^q}$. Since $z(t) \le z(t) + \mu(t) h z^{\Delta}(t)$ for $0 \le h \le 1$, and $t \in \mathbb{T}^{\kappa}_{[t_0,T]}$, we have

$$\varphi^{-1}(z(t)) \le \varphi^{-1}\left(z(t) + \mu(t) h z^{\Delta}(t)\right), \quad t \in \mathbb{T}^{\kappa}_{[t_0,T]}.$$

Hence from (2.5), we get

$$\begin{aligned} (G(z(t)))^{\Delta} &= z^{\Delta}(t) \int_0^1 G'\left((z(t) + \mu(t) h z^{\Delta}(t)\right) dh \\ &= z^{\Delta}(t) \int_0^1 \frac{1}{[\varphi^{-1}(z(t) + \mu(t) h z^{\Delta}(t))]^q} dh \\ &\le z^{\Delta}(t) \int_0^1 \frac{1}{[\varphi^{-1}(z(t))]^q} dh \\ &= \frac{z^{\Delta}(t)}{[\varphi^{-1}(z(t))]^q}, \end{aligned} \tag{2.6}$$

for $t \in \mathbb{T}^{\kappa}_{[t_0,T]}$. Combining (2.4) and (2.6), we obtain

$$(G(z(t)))^{\Delta} \le \sum_{i=1}^{l} \left[f_i(t)\psi(\varphi^{-1}(z(t))) + g_i(t)\right], \quad t \in \mathbb{T}^{\kappa}_{[t_0,T]}. \tag{2.7}$$

Integrating (2.7) from t_0 to t and taking into account that $g_i : \mathbb{T}_{[t_0,T]} \mapsto \mathbb{R}_0$ for each i, we deduce

$$G(z(t)) \le G(\varepsilon + b(T)) + \sum_{i=1}^{l} \int_{t_0}^{t} f_i(s)\psi(\varphi^{-1}(z(s)))\Delta s + \sum_{i=1}^{l} \int_{t_0}^{T} g_i(s)\Delta s. \tag{2.8}$$

for all $t \in \mathbb{T}_{[t_0,T]}$. Now define a function $v(t)$ by the right–hand side of (2.8), that is,

$$0 < v(t) = G(\varepsilon + b(T)) + \sum_{i=1}^{l} \int_{t_0}^{T} g_i(s)\Delta s + \sum_{i=1}^{l} \int_{t_0}^{t} f_i(s)\psi(\varphi^{-1}(z(s)))\Delta s.$$

Clearly, $v(t)$ is nondecreasing, $z(t) \leq G^{-1}(v(t))$ for $t \in \mathbb{T}_{[t_0,T]}$ and

$$v(t_0) = G(\varepsilon + b(T)) + \int_{t_0}^{T} \sum_{i=1}^{l} g_i(s)\Delta s.$$

Therefore, for any $t \in [t_0,T]_{\mathbb{T}}^{\kappa}$, we have

$$\begin{aligned} v^{\Delta}(t) &= \sum_{i=1}^{l} f_i(t)\psi(\varphi^{-1}(z(t))) \\ &\leq \psi(\varphi^{-1}(G^{-1}(v(t)))) \sum_{i=1}^{l} f_i(t). \end{aligned}$$

Using the monotonicity of ψ, φ^{-1}, G^{-1}, and v yields

$$\frac{v^{\Delta}(t)}{\psi(\varphi^{-1}(G^{-1}(v(t))))} \leq \sum_{i=1}^{l} f_i(t), \quad t \in \mathbb{T}_{[t_0,T]}^{\kappa}. \tag{2.9}$$

On the other hand, as before, taking into account Lemma 1.1 and the definition of Ω, we have

$$\begin{aligned} (\Omega(v(t)))^{\Delta} &= (\Omega \circ v)^{\Delta}(t) \\ &= v^{\Delta}(t) \int_{0}^{1} \Omega'(v(t) + \mu(t)hv^{\Delta}(t))dh, \end{aligned} \tag{2.10}$$

for $t \in \mathbb{T}_{[t_0,T]}^{\kappa}$, where $\Omega'(t) = \frac{1}{\psi(\varphi^{-1}(G^{-1}(t)))}$. Since $v(t) \leq v(t) + \mu(t)hv^{\Delta}(t)$ for $0 \leq h \leq 1$, $t \in \mathbb{T}_{[t_0,T]}^{\kappa}$, we have

$$\psi\left(\varphi^{-1}\left(G^{-1}(v(t))\right)\right) \leq \psi\left(\varphi^{-1}\left(G^{-1}(v(t) + \mu(t)v^{\Delta}(t))\right)\right), \quad t \in \mathbb{T}_{[t_0,T]}^{\kappa}$$

and so

$$\frac{1}{\psi\left(\varphi^{-1}\left(G^{-1}(v(t) + \mu(t)hv^{\Delta}(t))\right)\right)} \leq \frac{1}{\psi\left(\varphi^{-1}\left(G^{-1}(v(t))\right)\right)}, \quad t \in \mathbb{T}_{[t_0,T]}^{\kappa}.$$

Substituting this last inequality into (2.10) and taking into account (2.9) we obtain

$$(\Omega(v(t)))^{\Delta} \leq \frac{v^{\Delta}(t)}{\psi\left(\varphi^{-1}\left(G^{-1}(v(t))\right)\right)} \leq \sum_{i=1}^{l} f_i(t), \qquad t \in \mathbb{T}_{[t_0,T]}^{\kappa}. \tag{2.11}$$

Integrating (2.11) from t_0 to t yields

$$\begin{aligned}\Omega(v(t)) &\leq \Omega(v(t_0)) + \int_{t_0}^{t} \left(\sum_{i=1}^{l} f_i(s)\right) \Delta s \\ &= \Omega\left(G(\varepsilon + b(T)) + \int_{t_0}^{T} \sum_{i=1}^{l} g_i(s)\Delta s\right) + \int_{t_0}^{t} \left(\sum_{i=1}^{l} f_i(s)\right) \Delta s,\end{aligned}$$

for $t \in \mathbb{T}_{[t_0,T]}$ and hence we obtain

$$v(t) \leq \Omega^{-1}\left[\Omega\left(G(\varepsilon + b(T)) + \int_{t_0}^{T} \sum_{i=1}^{l} g_i(s)\Delta s\right) + \int_{t_0}^{t} \left(\sum_{i=1}^{l} f_i(s)\right) \Delta s\right],$$

for $t \in \mathbb{T}_{[t_0,T]}$. Since $z(t) \leq G^{-1}(v(t))$ for $t \in \mathbb{T}_{[t_0,T]}$, we get

$$z(t) \leq G^{-1}\left\{\Omega^{-1}\left[\Omega\left(G(\varepsilon + b(T)) + \int_{t_0}^{T} \sum_{i=1}^{l} g_i(s)\Delta s\right) + \int_{t_0}^{t} \left(\sum_{i=1}^{l} f_i(s)\right) \Delta s\right]\right\}$$

for $t \in \mathbb{T}_{[t_0,T]}$. Letting $\varepsilon \to 0$ and taking into account that $u(t) \leq \varphi^{-1}(z(t))$ for $t \in \mathbb{T}_{[t_0,T]}$ we obtain

$$u(t) \leq \varphi^{-1}\left[G^{-1}\left\{\Omega^{-1}\left[\Omega\left(G(b(T)) + \int_{t_0}^{T} \sum_{i=1}^{l} g_i(s)\Delta s\right) + \int_{t_0}^{t} \left(\sum_{i=1}^{l} f_i(s)\right) \Delta s\right]\right\}\right].$$

The last inequality produces the required inequality (2.1) for $T = t$, since $T \in \mathbb{T}_{t_0}$ was arbitrary, which completes the proof. □

The following corollary follows from Theorem 2.1 when $\varphi(u) = u^p$, $G(r) = r^{\frac{p-q}{p}}$, where $p > q \geq 0$ are constants.

Corollary 2.2. *Let b, f_i, g_i, α, and ψ be as defined in Theorem 2.1. Suppose that $p > q \geq 0$ are constants. If $u : \mathbb{T}_a \mapsto \mathbb{R}_0$ and*

$$u^p(t) \leq b(t) + \sum_{i=1}^{l} \int_{t_0}^{t} u^q(\alpha(s)) \left[f_i(s)\psi(u(\alpha(s))) + g_i(s)\right] \Delta s$$

for $t \in \mathbb{T}_{t_0}$, then

$$u(t) \leq \left\{\Omega^{-1}\left[\Omega\left([b(t)]^{\frac{p-q}{p}} + \frac{p-q}{p}\int_{t_0}^{t} \sum_{i=1}^{l} g_i(s)\Delta s\right) + \frac{p-q}{p}\int_{t_0}^{t} \sum_{i=1}^{l} f_i(s)\Delta s\right]\right\}^{\frac{1}{p-q}}$$

for $t \in \mathbb{T}_{[t_0,\beta]}$, where

$$\Omega(r) = \int_{r_0}^{r} \frac{ds}{\psi\left[s^{\frac{1}{p-q}}\right]}, \quad r \geq r_0 > 0,$$

and Ω^{-1} denotes the inverse function of Ω and $\beta \geq t_0$ is chosen such that

$$\Omega\left([b(t)]^{\frac{p-q}{p}} + \frac{p-q}{p}\int_{t_0}^{t}\sum_{i=1}^{l} g_i(s)\Delta s\right) + \frac{p-q}{p}\int_{t_0}^{t}\sum_{i=1}^{l} f_i(s)\Delta s \in Dom\left(\Omega^{-1}\right)$$

holds.

Proof. The argument of the proof is same as in the proof of Theorem 2.1 with suitable modification. Hence we omit the details here. □

Remark 2.3. Let $\mathbb{T} = \mathbb{Z}$. If $p = 2$, $q = 1$, $b(t) = c^2$, $\alpha(s) = s$, $i = 1$ in Corollary 2.2, then our results deduces to the Pachpatte inequality in [9].

We finish this section with another useful nonlinear integral inequality without its proof since it is similar to the proof of Theorem 2.1. The approach is again based on a chain rule on time scales.

Theorem 2.4. *Let b, f_i, g_i, $i = 1, \cdots, l$, q, and φ be as defined in Theorem 2.1. Suppose that $\psi_j(u)$ $(j = 1, 2)$ is a nondecreasing continuous function for $u \in \mathbb{R}_0$ with $\psi_j(u) > 0$ for $u > 0$. If $u : \mathbb{T}_a \mapsto \mathbb{R}_1$ and*

$$\varphi(u(t)) \leq b(t) + \sum_{i=1}^{l}\int_{t_0}^{t} u^q(\alpha(s))\,[f_i(s)\psi_1(u(\alpha(s))) + g_i(s)\psi_2(\log u(\alpha(s)))]\,\Delta s$$

for $t \in \mathbb{T}_{t_0}$, then

- *for the case $\psi_1(u) \geq \psi_2(\log(u))$, we have*

$$u(t) \leq \varphi^{-1}\left\{G^{-1}\left[\Omega_1^{-1}\left(\Omega_1[G(b(t))] + \int_{t_0}^{t}\sum_{i=1}^{l}[f_i(s) + g_i(s)]\Delta s\right)\right]\right\}$$

  for $t \in \mathbb{T}_{[t_0,\beta_1]}$, and

- *for the case $\psi_1(u) < \psi_2(\log(u))$, we have*

$$u(t) \leq \varphi^{-1}\left\{G^{-1}\left[\Omega_2^{-1}\left(\Omega_2[G(b(t))] + \int_{t_0}^{t}\sum_{i=1}^{l}[f_i(s) + g_i(s)]\Delta s\right)\right]\right\}$$

  for $t \in \mathbb{T}_{[t_0,\beta_2]}$,

where

$$\Omega_m(r) = \int_{r_0}^{r}\frac{ds}{\psi_{m-1}(\varphi^{-1}(G^{-1}(s)))}, \quad r \geq r_0 > 0,$$

G^{-1}, and Ω_m^{-1}, $m = 1, 2$ denote the inverse functions of G, Ω_m, respectively, $G(t)$ is as defined in Theorem 2.1 for $t \in \mathbb{T}_{t_0}$, and $\beta_m \geq t_0$, $m = 1, 2$ is chosen such that

$$\Omega_m[G(b(t))] + \int_{t_0}^{t}\sum_{i=1}^{l}[f_i(s) + g_i(s)]\Delta s \in Dom\left(\Omega_m^{-1}\right)$$

holds.

The following corollary follows from Theorem 2.4 when $\varphi(u) = u^p$, $G(r) = r^{\frac{p-q}{p}}$, where $p > q \geq 0$ are constants.

Corollary 2.5. *Let b, f_i, g_i for $i = 1,2,\ldots,l$, ψ_j, $j = 1,2$ and α be as defined in Theorem 2.4. Suppose that $p > q \geq 0$ are constants. If $u : \mathbb{T}_a \mapsto \mathbb{R}_1$ and*

$$u^p(t) \leq b(t) + \sum_{i=1}^{l} \int_{t_0}^{t} u^q(\alpha(s)) \left[f_i(s)\psi_1(u(\alpha(s))) + g_i(s)\psi_2(\log u(\alpha(s)))\right] \Delta s$$

for $t \in \mathbb{T}_{t_0}$, then

- *for the case $\psi_1(u) \geq \psi_2(\log(u))$, we have*

$$u(t) \leq \left\{ \Omega_1^{-1} \left[\Omega_1 \left([b(t)]^{\frac{p-q}{p}} \right) + \frac{p-q}{p} \int_{t_0}^{t} \sum_{i=1}^{l} [f_i(s) + g_i(s)] \Delta s \right] \right\}^{\frac{1}{p-q}}$$

  for $t \in \mathbb{T}_{[t_0,\beta_1]}$, and

- *for the case $\psi_1(u) < \psi_2(\log(u))$, we have*

$$u(t) \leq \left\{ \Omega_2^{-1} \left[\Omega_2 \left([b(t)]^{\frac{p-q}{p}} \right) + \frac{p-q}{p} \int_{t_0}^{t} \sum_{i=1}^{l} [f_i(s) + g_i(s)] \Delta s \right] \right\}^{\frac{1}{p-q}}$$

  for $t \in \mathbb{T}_{[t_0,\beta_2]}$,

where

$$\Omega_m(r) = \int_{r_0}^{r} \frac{ds}{\psi_{m-1}(\varphi^{-1}(G^{-1}(s)))}, \quad r \geq r_0 > 0,$$

G^{-1}, and Ω_m^{-1}, $m = 1,2$ denote the inverse functions of G, Ω_m, respectively, $G(t)$ is as defined in Theorem 2.1 for $t \in \mathbb{T}_{t_0}$, and $\beta_m \geq t_0$, $m = 1,2$ is chosen such that

$$\Omega_m \left([b(t)]^{\frac{p-q}{p}} \right) + \frac{p-q}{p} \int_{t_0}^{t} \sum_{i=1}^{l} [f_i(s) + g_i(s)] \Delta s \in Dom\left(\Omega_m^{-1}\right)$$

holds.

3 Applications

Our results are helpful in proving the global existence of solutions to certain dynamic equations with time delay. We first consider the functional dynamic equation

$$\begin{cases} \phi^{\Delta}(x(t)) = h(t) + \sum\limits_{i=1}^{l} F_i\left[t, x(\alpha(t)), w(x(\alpha(t)))\right] \\ \phi(x(t_0)) = x_0 \end{cases}, \tag{3.1}$$

where x_0 is a constant, $\phi \in C(\mathbb{R}, \mathbb{R}_0)$ is an increasing function such that $\phi(|x|) \leq |\phi(x)|$, $h : \mathbb{T}_{t_0} \mapsto \mathbb{R}$ is nondecreasing, $x : \mathbb{T}_a \mapsto \mathbb{R}$, $\alpha : \mathbb{T}_{t_0} \mapsto \mathbb{T}$ is nondecreasing such that $\alpha(t) \leq t$ and $-\infty < a = \inf\{\alpha(s) : s \in \mathbb{T}_{t_0}\}$, $w \in C(\mathbb{R}, \mathbb{R})$ is a nondecreasing function, and $F_i \in C(\mathbb{T}_{t_0} \times \mathbb{R}^2, \mathbb{R})$.

The following theorem deals with the bound on the solution of (3.1).

Theorem 3.1. *Assume that $F_i : \mathbb{T}_{t_0} \times \mathbb{R}^2 \to \mathbb{R}$ for $i = 1, \ldots, l$ is a continuous function and there exist continuous functions f_i, $g_i \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}_{t_0}, \mathbb{R}_0)$, $i = 1, \ldots, l$ such that*

$$|F_i[t, x(\alpha(t)), w(x(\alpha(t)))]| \leq |x(\alpha(t))|^q f_i(t)\psi(|x(\alpha(t))|) + g_i(t), \tag{3.2}$$

and

$$|x_0| + \int_{t_0}^{t} |h(s)|\Delta s \leq b(t), \tag{3.3}$$

where $q \geq 0$ is a constant and $b(t)$, ψ are as in Theorem 2.1. If $x(t)$ is any solution of (3.1) for $t \in \mathbb{T}_{t_0}$, then

$$|x(t)| \leq \phi^{-1}\left\{G^{-1}\left[\Omega^{-1}\left(\Omega\left(G(b(t) + \int_{t_0}^{t}\sum_{i=1}^{l} g_i(s)\Delta s\right) + \int_{t_0}^{t}\sum_{i=1}^{l} f_i(s)\Delta s\right)\right]\right\} \tag{3.4}$$

for $t \in \mathbb{T}_{t_0}$, where G and Ω are defined as in Theorem 2.1.

Proof. Let $x(t)$ be a solution of (3.1) for $t \in \mathbb{T}_{t_0}$. One can show that $x(t)$ satisfies the equivalent equation

$$\phi(x(t)) = x_0 + \int_{t_0}^{t} h(s)\Delta s + \sum_{i=1}^{l}\int_{t_0}^{t} F_i[s, x(\alpha(s)), w(x(\alpha(s)))]\Delta s, \tag{3.5}$$

for $t \in \mathbb{T}_{t_0}$. It follows from (3.5) that

$$|\phi(x(t))| \leq |x_0| + \int_{t_0}^{t} |h(s)|\Delta s + \sum_{i=1}^{l}\int_{t_0}^{t} |F_i[s, x(\alpha(s)), w(x(\alpha(s)))]|\Delta s \tag{3.6}$$

for $t \in \mathbb{T}_{t_0}$. Using the conditions (3.2), (3.3) on the right hand side of (3.6) we obtain

$$\phi(|x(t)|) \leq b(t) + \sum_{i=1}^{l}\int_{t_0}^{t} [|x(\alpha(s))|^q f_i(s)\psi(|x(\alpha(s))|) + g_i(s)]\Delta s$$

for $t \in \mathbb{T}_{t_0}$. Now an immediate application of the inequality established in Theorem 2.1 to the inequality (3.4) yields the result. □

Remark 3.2. We now consider the functional dynamic equation with the initial condition

$$\begin{cases} (x^p(t))^\Delta = h(t) + \sum\limits_{i=1}^{l} F_i[t, x(\alpha(t)), w(x(\alpha(t)))], \\ x^p(t_0) = x_1 \end{cases} \tag{3.7}$$

where $p > 0, x_1$ are constants. Assume that $F_i : \mathbb{T}_{t_0} \times \mathbb{R}^2 \mapsto \mathbb{R}$ for $i = 1, \ldots, l$ is a continuous function and there exist continuous functions $f_i,\ g_i : \mathbb{T}_{t_0} \mapsto \mathbb{R}_0,\ i = 1, \ldots, l$ such that the inequalities (3.2) and (3.3) hold, where $q \geq 0$ is a constant such that $p > q$ and $b(t)$, ψ are defined as in Corollary 2.2. If $x(t)$ is any solution of the problem (3.7) for $t \in \mathbb{T}_{t_0}$, then it satisfies the equivalent equation

$$x^p(t) = x_1 + \int_{t_0}^{t} h(s)\Delta s + \sum_{i=1}^{l} \int_{t_0}^{t} F_i\left[s, x(\alpha(s)), w(x(\alpha(s)))\right]\Delta s, \tag{3.8}$$

for $t \in \mathbb{T}_{t_0}$. It follows from (3.8) that

$$|x(t)|^p \leq |x_1| + \int_{t_0}^{t} |h(s)|\Delta s + \sum_{i=1}^{l} \int_{t_0}^{t} \left|F_i\left[s, x(\alpha(s)), w(x(\alpha(s)))\right]\right|\Delta s \tag{3.9}$$

for $t \in \mathbb{T}_{t_0}$. Using the conditions (3.2), (3.3) on the right hand side of (3.9) yields

$$|x(t)|^p \leq b(t) + \sum_{i=1}^{l} \int_{t_0}^{t} \left[|x(\alpha(s))|^q f_i(s)\psi(|x(\alpha(s))|) + g_i(t)\right]\Delta s, \tag{3.10}$$

where $t \in \mathbb{T}_{t_0}$. Now an immediate application of the inequality established in Corollary 2.2 to (3.10) yields

$$|x(t)| \leq \left\{\Omega^{-1}\left[\Omega\left([b(t)]^{\frac{p-q}{p}} + \frac{p-q}{p}\int_{t_0}^{t}\sum_{i=1}^{l} g_i(s)\Delta s\right) + \frac{p-q}{p}\int_{t_0}^{t}\sum_{i=1}^{l} f_i(s)\Delta s\right]\right\}^{\frac{1}{p-q}}$$

for $t \in \mathbb{T}_{t_0}$, where Ω is as in Corollary 2.2.

In the following theorem we give necessary conditions to obtain a unique solution of (3.7).

Theorem 3.3. *Assume that $F_i : \mathbb{T}_{t_0} \times \mathbb{R}^3 \to \mathbb{R}$ for $i = 1, \ldots, l$ is a continuous function and there exists a continuous nonnegative function $f_i(t)$ for $i = 1, \ldots, l$ for $t \in \mathbb{T}_{t_0}$ such that*

$$|F(t, x, w(x)) - F(t, \bar{x}, w(\bar{x}))| \leq f_i(t)\,|x^p - \bar{x}^p|, \tag{3.11}$$

where $p > 1$ is a constant, then the problem (3.7) has a unique solution on $\mathbb{T}_{t_0}$.

Proof. Let $x(t)$ and $\bar{x}(t)$ be two solutions of (3.7) for $t \in \mathbb{T}_{t_0}$. Then we have

$$x^p(t) - \bar{x}^p(t) = \sum_{i=1}^{l} \int_{t_0}^{t} \left[F_i\left[s, x(\alpha(s)), w(x(\alpha(s)))\right] - F_i\left[s, \bar{x}(\alpha(s)), w(\bar{x}(\alpha(s)))\right]\right]\Delta s, \tag{3.12}$$

for $t \in \mathbb{T}_{t_0}$. From (3.11) and (3.12), we get

$$|x^p(t) - \bar{x}^p(t)| \leq \sum_{i=1}^{l} \int_{t_0}^{t} f_i(s)\,|x^p(\alpha(s)) - \bar{x}^p(\alpha(s))|\,\Delta s \tag{3.13}$$

for $t \in \mathbb{T}_{t_0}$. Rearranging equation (3.13) yields

$$\left(|x^p(t) - \bar{x}^p(t)|^{\frac{1}{p}}\right)^p \leq \sum_{i=1}^{l} \int_{t_0}^{t} \left[|x^p(\alpha(s)) - \bar{x}^p(\alpha(s))|^{\frac{1}{p}}\right]^{p-1} f_i(s) \left[|x^p(\alpha(s)) - \bar{x}^p(\alpha(s))|^{\frac{1}{p}}\right] \Delta s, \tag{3.14}$$

where $t \in \mathbb{T}_{t_0}$ when $\psi(u) = u$, $q = p-1$, a suitable application of the inequality in Corollary 2.2 to the function $|x^p(t) - \bar{x}^p(t)|^{\frac{1}{p}}$ and the inequality (3.14) lead us to the inequality

$$|x^p(t) - \bar{x}^p(t)|^{\frac{1}{p}} \leq 0$$

for all $t \in \mathbb{T}_{t_0}$. Hence we obtain $x(t) = \bar{x}(t)$ for $t \in \mathbb{T}_{t_0}$. □

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