Nonlinear Dynamics and Systems Theory, 14 (3) (2014) 208–222

Almost Oscillatory Three-Dimensional Dynamical Systems of First Order Delay Dynamic Equations

A. Akgül [1,2] and E. Akın [2*]

[1] *Dicle University, Education Faculty Department of Mathematics 21280 Diyarbakır, Turkey,*
[2] *Missouri University of Science and Technology Department of Mathematics and Statistics Rolla, Missouri 65409-0020, USA*

Received: February 17, 2014;   Revised: July 5, 2014

Abstract: In this paper, we investigate oscillation and asymptotic properties for three dimensional systems of first order dynamic equations with delays. Most of our results are new in the discrete case.

Keywords: *time scales; oscillation; three-dimensional dynamical system.*

Mathematics Subject Classification (2010): 39A10.

1 Introduction

In this paper, we investigate three dimensional dynamical systems with delays of the form

$$\begin{cases} x^{\Delta}(t) = a(t)f(y(\tau(t))), \\ y^{\Delta}(t) = b(t)g(z(\tau(t))), \\ z^{\Delta}(t) = \lambda\, c(t)h(x(\tau(t))), \end{cases} \tag{1}$$

on a time scale $\mathbb{T}$, i.e, a closed subset of real numbers, $\tau : \mathbb{T} \to \mathbb{T}$ is a rd-continuous function such that $\tau(t) < t$, $\lim_{t\to\infty} \tau(t) = \infty$, $\lambda = \pm 1$, $a, b : \mathbb{T} \mapsto [0, \infty)$ (not identically zero) and $c : \mathbb{T} \mapsto (0, \infty)$ are rd-continuous functions such that

$$\int_T^\infty a(s)\Delta s = \int_T^\infty b(s)\Delta s = \infty, \quad T \in \mathbb{T} \tag{2}$$

and $f, g, h : \mathbb{R} \mapsto \mathbb{R}$ are continuous functions satisfying

$$uf(u) > 0, \quad ug(u) > 0, \quad \text{and} \quad uh(u) > 0 \quad \text{for } u \neq 0. \tag{3}$$

* Corresponding author: mailto:akine@mst.edu

Here, we would like to indicate that none of the functions f, g and h are assumed to be monotone. Sometimes we will assume that functions f, g and h satisfy

$$\frac{f(u)}{\Phi_\alpha(u)} \geq F, \quad \frac{g(u)}{\Phi_\beta(u)} \geq G, \quad \frac{h(u)}{\Phi_\gamma(u)} \geq H \quad \text{for all } u \neq 0, \tag{4}$$

where F, G, H are positive constants and Φ_α, Φ_β and Φ_γ are odd power functions, i.e.

$$\Phi_p(u) = |u|^p \operatorname{sgn} u \quad (p > 0), \qquad p \in \{\alpha, \beta, \gamma\}.$$

This paper is motivated by the papers [1,2,6]. In [1], the special case of system (1) has been considered in which $f(u) = u^\alpha$, $g(u) = u^\beta$, $h(u) = u^\gamma$, $\tau(t) = t$, $\lambda = -1$, and α, β, γ are ratios of odd positive integers. In [2], system (1) is considered without delays. The continuous version of a system similar to system (1) without delays in [5] and the discrete version of a system similar to system (1) with delays in [6,7] have been considered. The results in [8] are the discrete version of these in [1]. It is worth mentioning that our results not only improve results in [6] but also are new in the discrete case.

The main purpose of this paper is to investigate oscillatory and asymptotic behaviour of solutions of system (1). The set up in this paper is as follows: In Section 2, we give preliminary results including some asymptotic behaviour of the solutions of system (1). In Sections 3 and 4, we obtain almost oscillation criteria for solutions of system (1) when $\lambda = -1$ and $\lambda = 1$, respectively.

Here, we consider only unbounded time scales. For an excellent introduction to time scales we refer the interested reader to the books [3,4].

A proper solution of system (1) is said to be oscillatory if all its components x, y, z are oscillatory. System (1) with $\lambda = 1$ is said to be almost oscillatory if every solution (x, y, z) of system (1) is either oscillatory or

$$\lim_{t\to\infty} | x(t) | = \lim_{t\to\infty} | y(t) | = \lim_{t\to\infty} | z(t) | = \infty. \tag{5}$$

System (1) with $\lambda = -1$ is said to be almost oscillatory if every solution (x, y, z) of system (1) is either oscillatory or

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = \lim_{t\to\infty} z(t) = 0. \tag{6}$$

It is necessary to use the following remark in the further sections in order to obtain a contradiction.

Remark 1.1 (See [1]) Let $a, c \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ such that $\int_T^\infty c(s)\Delta s < \infty$. Then

$$\int_T^\infty a(t)\left(\int_t^\infty c(s)\Delta s\right)\Delta t = \int_T^\infty c(t)\left(\int_T^{\sigma(t)} a(s)\Delta s\right)\Delta t.$$

2 Preliminaries

In this section, we investigate asymptotic behaviour of solutions of system (1) so that we will be able to obtain almost oscillatory systems. The next two results hold regardless if $\lambda = \pm 1$. In the following subsections, we will classify nonoscillatory solutions of system (1) when $\lambda = 1$ and $\lambda = -1$, respectively.

Lemma 2.1 *Assume that condition* (3) *holds. Let* (x, y, z) *be a solution of system* (1) *and let* $x(t)$ *be nonoscillatory for* $t \geq t_0$, $t_0 \in \mathbb{T}$. *Then* (x, y, z) *is nonoscillatory and* x,y,z *are monotonic for sufficiently large* t.

Proof. Let (x, y, z) be a solution of system (1) such that $x(t)$ is nonoscillatory for $t \geq t_0$. Then we assume that $x(\tau(t)) > 0$ *for* $t \geq t_1 \geq t_0$, $t_1 \in \mathbb{T}$. By the third equation of system (1), we have $z^{\Delta}(t) > 0$ or $z^{\Delta}(t) < 0$, $t \geq t_1 \geq t_0$. This implies that $z(t)$ is monotonic for $t \geq t_1 \geq t_0$ and eventually of one sign for $t \geq t_1$. Let $z(t) > 0$, $z(\tau(t)) > 0$ *for* $t \geq t_2 \geq t_1$, $t_2 \in \mathbb{T}$. Therefore from the second equation of system (1), $y(t)$ is monotonic for $t \geq t_2 \geq t_1$ and eventually of one sign for $t \geq t_2 \geq t_1$. Let $y(\tau(t)) > 0$ *for* $t \geq t_3 \geq t_2$. Similarly, we obtain that x(t) is monotonic for $t \geq t_3 \geq t_2$ from the first equation of system (1). Therefore (x, y, z) is nonoscillatory.

Lemma 2.2 *Assume that conditions* (2) *and* (3) *hold. Let* (x, y, z) *be a nonoscillatory solution of system* (1) *such that* $\lim_{t\to\infty} x(t)$ *is finite, then*

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} z(t) = 0.$$

Proof. Assume that (x, y, z) is a nonoscillatory solution of system (1) such that the limit of x is finite. By Lemma 2.1, y is monotonic and hence the limit of y exists. For the sake of contradiction suppose that the limit of y is positive. Therefore, $y(t) > 0$ for large t. Then there exists $t_1 \geq t_0$, $t_1 \in \mathbb{T}$ such that

$$y(\tau(t)) > 0, \quad \tau(t) \geq t_1.$$

From (3), there exist a positive constant K and $t_2 \in \mathbb{T}$, $t_2 \geq t_1$ such that

$$f(y(\tau(t))) > K, \quad \tau(t) \geq t_2.$$

Thus, from the first equation of system (1), we have

$$x^{\Delta}(t) \;=\; a(t)f(y(\tau(t))) > a(t)K > 0, \quad \tau(t) \geq t_2.$$

Integrating the above inequality from t_2 to t, we get

$$x(t) > x(t_2) + K\int_{t_2}^{t} a(s)\Delta s.$$

It follows from (2) that

$$\lim_{t\to\infty} x(t) = \infty,$$

but this gives us a contradiction. In the case the limit of y is negative, the proof is similar and hence omitted. Therefore, we get

$$\lim_{t\to\infty} y(t) = 0.$$

Similarly one can show that

$$\lim_{t\to\infty} z(t) = 0$$

by using the second equation of system (1). So this completes the proof.

2.1 Preliminaries when $\lambda = 1$

In this subsection, we will investigate asymptotic behaviour of solutions of system (1) when $\lambda = 1$.

Lemma 2.3 *Let conditions* (2) *and* (3) *hold. Assume that* (x, y, z) *is a nonoscillatory solution of system* (1) *with* $\lambda = 1$ *for large* t *and let*

$$\begin{aligned} &\text{Type (a):} \quad sgn\, x(t) = sgn\, y(t) = sgn\, z(t), \\ &\text{Type (c):} \quad sgn\, x(t) = sgn\, y(t) \neq sgn\, z(t). \end{aligned}$$

Then every nonoscillatory solution of system (1) *with* $\lambda = 1$ *is of either Type (a) or Type (c).*

Proof. Let (x, y, z) be a nonoscillatory solution of system (1). Without loss of generality, we assume that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \geq t_0$, $t_0 \in \mathbb{T}$. By Lemma 2.1, both y and z are monotonic. Therefore they are eventually of one sign. First let $z(t) > 0$ and $z(\tau(t)) > 0$ for $t \geq t_0$. Suppose $y(t) < 0$ for $t \geq t_0$. Since y is increasing, $y(\tau(t)) < 0$ for $t \geq t_0$. Since z is increasing, there exist $t_1 \in \mathbb{T}$ and $L > 0$ such that

$$g(z(\tau(t))) > L, \quad \tau(t) \geq t_1. \tag{7}$$

Using (7) and the second equation of system (1) yields

$$y^{\Delta}(t) \;=\; b(t)g(z(\tau(t))) > Lb(t), \quad \tau(t) \geq t_1.$$

If we integrate the above inequality from t_1 to t, we obtain

$$y(t) > y(t_1) + L \int_{t_1}^{t} b(s)\Delta s.$$

By (2), $y(t) \to \infty$ as $t \to \infty$, which is a contradiction with the negativity of y. Therefore this case is not possible and so (x, y, z) is of Type (a).

Now let $z(t) < 0$ for $t \geq t_0$. Since z is increasing, $z(\tau(t)) < 0$, $t \geq t_0$. Suppose that $y(t) < 0$, $y(\tau(t)) < 0$ for large t. Then there exist $t_1 \geq t_0$, $t_1 \in \mathbb{T}$ and $v \leq 0$ such that

$$f(y(\tau(t))) \leq v, \quad \tau(t) \geq t_1. \tag{8}$$

We claim that $v = 0$. Assume that $v < 0$ and we will show that this leads to a contradiction. Using (8) and the first equation of system (1) yields

$$x^{\Delta}(t) \;=\; a(t)f(y(\tau(t))) \leq va(t), \quad \tau(t) \geq t_1.$$

Integrating the last inequality from t_1 to t, we obtain

$$x(t) \leq x(t_1) + v \int_{t_1}^{t} a(s)\Delta s.$$

By (2), we get $x(t) \to -\infty$ *as* $t \to \infty$, which is a contradiction with the positivity of x. Therefore this case is not possible and so (x, y, z) is of Type (c).

The proof for the case when $x(t) < 0$ for large t is analogous.

Solutions of Type (a) are sometimes called strongly monotone solutions (see, e.g. [5]).

Lemma 2.4 *Let conditions* (2) *and* (3) *hold. Any Type (a) solution* (x, y, z) *of system* (1) *with* $\lambda = 1$ *satisfies*

$$\lim_{t\to\infty} | x(t) |= \lim_{t\to\infty} | y(t) |= \infty.$$

Proof. Let (x, y, z) be a Type (a) solution of system (1). Then there exists $t_0 \in \mathbb{T}$ such that $x(\tau(t)) > 0$, $y(\tau(t)) > 0$, and $z(\tau(t)) > 0$ for $t \geq t_0$. Since y is eventually increasing, there exist $t_1 \geq t_0$, $t_1 \in \mathbb{T}$ and $K > 0$ such that $f(y(\tau(t))) \geq K$, $\tau(t) \geq t_1$. From the first equation of system (1), we have

$$x^{\Delta}(t) = a(t)f(y(\tau(t))) \geq Ka(t), \quad \tau(t) \geq t_1.$$

Integrating the above inequality from t_1 to t yields

$$x(t) \geq x(t_1) + K\int_{t_1}^{t} a(s)\Delta s, \quad \tau(t) \geq t_1.$$

The above inequality together with (2) implies that $\lim_{t\to\infty} x(t) = \infty$. Since z is eventually increasing, there exist $t_2 \geq t_1$, $t_2 \in \mathbb{T}$ and $M > 0$ such that $g(z(\tau(t))) \geq M$, $\tau(t) \geq t_2$. From the second equation of system (1), we have

$$y^{\Delta}(t) = b(t)g(z(\tau(t))) \geq Mb(t), \quad \tau(t) \geq t_2.$$

Integrating the above inequality from t_2 to t gives us

$$y(t) \geq y(t_2) + M\int_{t_2}^{t} b(s)\Delta s, \quad \tau(t) \geq t_2. \tag{9}$$

The above inequality together with (2) implies $\lim_{t\to\infty} y(t) = \infty$. This completes the proof.

Lemma 2.5 *Let* (2) *and* (3) *hold. Assume that* (x, y, z) *is a Type (c) solution of system* (1) *with* $\lambda = 1$. *Then*

$$\lim_{t\to\infty} z(t) = 0.$$

Proof. Assume that (x, y, z) is a Type (c) solution of system (1). Without loss of generality, assume that $x(\tau(t)) > 0$ for $t \geq t_0$, $t_0 \in \mathbb{T}$. Then $y(t) > 0$, $z(t) < 0$, $t \geq t_0$. Since z is increasing, $\lim_{t\to\infty} z(t) \leq 0$. Suppose that $\lim_{t\to\infty} z(t) < 0$. Then there exist $t_1 \geq t_0$, $t_1 \in \mathbb{T}$ and $S < 0$ such that $g(z(\tau(t))) \leq S$, $\tau(t) \geq t_1$. Integrating the second equation of system (1) from t_1 to t, we have

$$y(t) \leq y(t_1) + S\int_{t_1}^{t} b(s)\Delta s, \quad \tau(t) \geq t_1$$

and therefore (2) implies that $\lim_{t\to\infty} y(t) = -\infty$. But this contradicts the fact that $y(t) > 0$ for $t \geq t_0$. Therefore, $\lim_{t\to\infty} z(t) = 0$. This completes the proof.

2.2 Preliminaries when $\lambda = -1$

In this subsection, we will investigate the asymptotic behaviour of solutions of system (1) when $\lambda = -1$.

Lemma 2.6 *Let conditions* (2) *and* (3) *hold. Then any nonoscillatory solution* (x, y, z) *of system* (1) *with* $\lambda = -1$ *is one of the following types:*

$$\begin{array}{ll} \text{Type (a):} & sgn\, x(t) = sgn\, y(t) = sgn\, z(t) \quad \textit{for} \quad \textit{large}\ t; \\ \text{Type (b):} & sgn\, x(t) = sgn\, z(t) \neq sgn\, y(t) \quad \textit{for} \quad \textit{large}\ t. \end{array}$$

Proof. Let (x, y, z) be a nonoscillatory solution of system (1). Without loss of generality, we assume that $x(t) > 0$, $x(\tau(t)) > 0$ for $t \geq t_0$. By Lemma 2.1, both y and z are monotonic and they are eventually of one sign. We now show that z cannot be negative. Suppose that $z(t) < 0$ for $t \geq t_0$ to obtain a contradiction. Then there exists $t_1 \geq t_0$, $t_1 \in \mathbb{T}$ such that $z(\tau(t)) < 0$ for $\tau(t) \geq t_1$. Then there exist $t_2 \in \mathbb{T}$, $t_2 \geq t_1$ and a constant $d \leq 0$ such that

$$g(z(\tau(t))) \leq d, \quad \tau(t) \geq t_2. \tag{10}$$

We claim that $d = 0$. Assume that $d < 0$ and we will show that this leads to a contradiction. If we use (10) together with the second equation of system (1), we obtain

$$y^{\Delta}(t) \quad = \quad b(t)g(z(\tau(t))) \leq db(t), \quad \tau(t) \geq t_2.$$

Integrating the above inequality from t_2 to t, we get

$$y(t) \leq y(t_2) + d \int_{t_2}^{t} b(s)\Delta s.$$

In view of (2), $y(t) \to -\infty$ *as* $t \to \infty$. Therefore, there exist $t_3 \in \mathbb{T}$, $t_3 \geq t_2$ and a negative constant v such that

$$y(\tau(t)) < v, \quad \tau(t) \geq t_3. \tag{11}$$

From (3) and (11), there exist $K < 0$ and $t_4 \in \mathbb{T}$, $t_4 \geq t_3$ such that

$$f(y(\tau(t))) \leq K, \quad \tau(t) \geq t_4. \tag{12}$$

Using (12) together with the first equation of system (1), we obtain

$$x^{\Delta}(t) \quad = \quad a(t)f(y(\tau(t))) \leq Ka(t), \quad \tau(t) \geq t_4.$$

If we integrate the last inequality from t_4 to t, we get

$$x(t) < x(t_4) + K \int_{t_4}^{t} a(s)\Delta s.$$

By (2), we have $x(t) \to -\infty$ as $t \to \infty$, but this contradicts the fact that $x(t) > 0$ for all $t \geq t_0$. This implies that $z(t) > 0$ for all $t \geq t_0$.

One can show the proof similarly for the case when $x(t) < 0$ eventually for $t \geq t_0$.

Lemma 2.7 *Let conditions* (2) *and* (3) *hold. Assume* (x, y, z) *is a Type (b) solution of system* (1) *with* $\lambda = -1$. *Then*

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} z(t) = 0.$$

Proof. Assume (x, y, z) is a Type (b) solution of system (1) such that $x(t) > 0$, $y(t) < 0$, $z(t) > 0$, $z(\tau(t)) > 0$ for $t \geq t_0$, $t_0 \in \mathbb{T}$. Since $y(t)$ is increasing, we have $\lim_{t\to\infty} y(t) \leq 0$. Assume $\lim_{t\to\infty} y(t) \neq 0$. Then there exist $t_1 \geq t_0$ and a constant $L < 0$ such that $y(\tau(t)) \leq L$ for $\tau(t) \geq t_1$. From (3), there exists $K < 0$ such that

$$f(y(\tau(t))) \leq K, \quad \tau(t) \geq t_1. \tag{13}$$

Integrating the first equation of system (1) from t_1 to t and using (13), we have

$$x(t) \leq x(t_1) + K \int_{t_1}^{t} a(s)\Delta s, \quad \tau(t) \geq t_1$$

and so (2) implies $\lim_{t\to\infty} x(t) = -\infty$. This contradicts the positivity of x and therefore $\lim_{t\to\infty} y(t) = 0$. In a similar way, we can show that $\lim_{t\to\infty} z(t) = 0$.

In the next two sections, we will obtain almost oscillation criteria for system (1).

3 Almost Oscillatory System (1) When $\lambda = -1$

The next two results in this section are new in the discrete case and can be found in [[2], Theorem 4.1, Theorem 4.2 and Theorem 4.3.] without delays.

Theorem 3.1 *Let conditions* (2) *and* (3) *hold. Assume*

$$\int_{T}^{\infty} c(s)\Delta s = \infty, \quad T \in \mathbb{T}. \tag{14}$$

Then system (1) *with* $\lambda = -1$ *is almost oscillatory.*

Proof. Assume (x, y, z) is a nonoscillatory solution of system (1). By Lemma 2.6, nonoscillatory solutions are of either Type (a) or Type (b). Assume (x, y, z) is a Type (a) solution. Without loss of generality, assume that there exists $t_0 \in \mathbb{T}$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $y(t) > 0$, $y(\tau(t)) > 0$, and $z(t) > 0$ for $t \geq t_0$. Since $x(t)$ is eventually increasing, there exist $L > 0$ and $t_1 \geq t_0$ such that $x(\tau(t)) > L$ for $\tau(t) \geq t_1$. From (3), there exist $K > 0$ and $t_2 \in \mathbb{T}$, $t_2 \geq t_1$ such that

$$h(x(\tau(t))) \geq K, \quad \tau(t) \geq t_2. \tag{15}$$

Integrating the third equation of system (1) from t_2 to t and using (15), we have

$$z(t) \geq z(t_2) + K \int_{t_2}^{t} c(s)\Delta s, \quad \tau(t) \geq t_2,$$

and so this implies that $\lim_{t\to\infty} z(t) = \infty$, which is a contradiction with the boundedness of z. Therefore, (x, y, z) can not be a Type (a) solution. Therefore all nonoscillatory solutions are of Type (b). Without loss of generality, assume that there exists $t_0 \in \mathbb{T}$ such that $x(t) > 0$, $y(t) < 0$, $y(\tau(t)) < 0$, $z(t) > 0$, $t \geq t_0$. By Lemma 2.7, we have $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} z(t) = 0$. So it is enough to show that $\lim_{t\to\infty} x(t) = 0$. Since x is eventually decreasing, there exists $t_1 \geq t_0$ such that $\lim_{t\to\infty} x(t) = M \geq 0$, $t \geq t_1$. Therefore there exists $t_2 \geq t_1$ such that $x(\tau(t)) \geq M$, $\tau(t) \geq t_2$. By (3), there exist $K > 0$ and $t_3 \geq t_2$ such that

$$h(x(\tau(t))) \geq K, \quad \tau(t) \geq t_3. \tag{16}$$

Integrating the third equation of system (1) from t_3 to t and using (16), we get

$$z(t) \leq z(t_3) - K \int_{t_3}^{t} c(s)\Delta s, \quad \tau(t) \geq t_3.$$

and as $t \to \infty$, we get a contradiction with the boundedness of z. So $\lim_{t\to\infty} x(t) = 0$. This completes the proof.

Example 3.1 Let $\mathbb{T} = \mathbb{Z}$. Then we consider the following system

$$\begin{cases} \Delta x_n = a_n f(y_{n-l}), \\ \Delta y_n = b_n g(z_{n-l}), \\ \Delta z_n = \lambda c_n h(y_{n-l}), \end{cases} \tag{17}$$

where l is a given positive integer and $\lambda = -1$. Here $a_n, b_n : \mathbb{N}_{n_0} \to \mathbb{R}_+ \cup \{0\}$, $c_n : \mathbb{N}_{n_0} \to \mathbb{R}_+$ such that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n = \infty, \tag{18}$$

where $n_0 \in \mathbb{N} = \{1, 2, \ldots\}$, R_+ is the set of positive real numbers. Also $f, g, h : \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying (3). If

$$\sum_{n=1}^{\infty} c_n = \infty, \tag{19}$$

then system (17) with $\lambda = -1$ is almost oscillatory by Theorem 3.1.

For the next two theorems, we assume that

$$\int_{T}^{\infty} c(s)\Delta s < \infty, \quad T \in \mathbb{T}. \tag{20}$$

Theorem 3.2 *Let $\lambda = -1$ in system (1). Assume condition (3) holds and there exist positive constants F, G, α, β such that*

$$\frac{f(u)}{\Phi_\alpha(u)} \geq F, \quad \frac{g(u)}{\Phi_\beta(u)} \geq G \quad \text{for small } u \neq 0. \tag{21}$$

If

$$\int_{T}^{\infty} b(s) \left(\int_{\tau(s)}^{\infty} c(v)\Delta v \right)^{\beta} \Delta s = \infty, \quad T \in \mathbb{T}, \tag{22}$$

or

$$\int_T^\infty a(t)\left(\int_{\tau(t)}^\infty b(s)\left(\int_{\tau(s)}^\infty c(v)\Delta v\right)^\beta \Delta s\right)^\alpha \Delta s=\infty,\quad T\in\mathbb{T}, \tag{23}$$

then every nonoscillatory solution of system (1) *that fulfils Type (b) satisfies* $\lim_{t\to\infty} x(t)=0$.

Proof. Assume that (x,y,z) is a nonoscillatory solution of system (1) of Type (b). Without loss of generality assume that $x(t)>0,\ y(t)<0,\ y(\tau(t))<0$ and $z(t)>0$ for $t\ge t_0$. From the first equation of system (1), x is nonincreasing, and therefore x has a nonnegative limit. Assume that $\lim_{t\to\infty} x(t)>0$. Then there exists $t_1\ge t_0$ such that $x(\tau(t))\ge 0,\ \tau(t)\ge t_1$. By (3), there exist $t_2\ge t_1$ and $K>0$ such that

$$h(x(\tau(t)))\ge K,\quad \tau(t)\ge t_2. \tag{24}$$

Integrating the third equation of system (1) from $\tau(t)$ to ∞ and using (24), we obtain

$$z(\tau(t))\ge K\int_{\tau(t)}^\infty c(s)\Delta s,\quad \tau(t)\ge t_2,$$

where we use Lemma 2.7. By (21) there exist $t_3\ge t_2,\ t_3\in\mathbb{T}$ and $G>0$ such that

$$g(z(\tau(t)))\ge GK^\beta\left(\int_{\tau(t)}^\infty c(s)\Delta s\right)^\beta,\quad \tau(t)\ge t_3. \tag{25}$$

Integrating the second equation of system (1) from t_3 to t and using (25), we obtain

$$\begin{aligned} y(t) &= y(t_3)+\int_{t_3}^t b(s)g(z(\tau(s)))\Delta s\\ &\ge y(t_3)+GK^\beta\int_{t_3}^t b(s)\left(\int_{\tau(s)}^\infty c(v)\Delta v\right)^\beta \Delta s,\quad \tau(t)\ge t_3. \end{aligned}$$

If we assume (22), then we have $\lim_{t\to\infty} y(t)=\infty$, but this contradicts the fact that $\lim_{t\to\infty} y(t)=0$. So $\lim_{t\to\infty} x(t)=0$. Assume (23). Integrating the second equation of system (1) from $\tau(t)$ to ∞ and using the fact that $\lim_{t\to\infty} y(t)=0$ and (25), we obtain

$$-y(\tau(t))\ge GK^\beta\int_{\tau(t)}^\infty b(s)\left(\int_{\tau(s)}^\infty c(v)\Delta v\right)^\beta \Delta s,\quad \tau(t)\ge t_3,$$

By (21), there exists $F>0$ such that

$$\begin{aligned} f(y(\tau(t))) &\le Fy^\alpha(\tau(t))\\ &\le -FG^\alpha K^{\alpha\beta}\left[\int_{\tau(t)}^\infty b(s)\left(\int_{\tau(s)}^\infty c(v)\Delta v\right)^\beta \Delta s\right]^\alpha,\quad \tau(t)\ge t_3. \end{aligned}$$

Integrating the first equation of system (1) from t_3 to t yields

$$\begin{aligned} x(t)-x(t_3) &= \int_{t_3}^{t} a(s)f(y(\tau(s)))\Delta s \\ &\leq -FG^{\alpha}K^{\alpha\beta}\int_{t_3}^{t} a(s)\left[\int_{\tau(s)}^{\infty} b(v)\left(\int_{\tau(v)}^{\infty} c(\eta)\Delta\eta\right)^{\beta}\Delta v\right]^{\alpha}\Delta s, \quad \tau(t)\geq t_3. \end{aligned}$$

This implies that $\lim_{t\to\infty} x(t) = -\infty$, which is a contradiction by (23). This completes the proof.

Example 3.2 Let $\mathbb{T}=\mathbb{Z}$. Then we consider system (17) with $\lambda=-1$. Assume there exist positive constants F, G, α, β such that (21) holds. If

$$\sum_{i=1}^{\infty} b_i\left(\sum_{r=i-l}^{\infty} c_r\right)^{\beta} = \infty$$

or

$$\sum_{i=1}^{\infty} a_i\left(\sum_{s=i-l}^{\infty} b_s\left(\sum_{r=s-l}^{\infty} c_r\right)^{\beta}\right)^{\alpha} = \infty$$

holds, then every nonoscillatory solution of system (17) that fulfils Type (b) satisfies $\lim_{t\to\infty} x(t) = 0$ by Theorem 3.2.

Theorem 3.3 *Assume conditions* (2), (3) *and* (4) *hold. Let* $\alpha\beta\gamma<1$. *If*

$$\int_{t_3}^{\infty} c(t)\left(\int_{t_2}^{\tau(t)} a(s)\left(\int_{t_1}^{\tau(s)} b(v)\Delta v\right)^{\alpha}\Delta s\right)^{\gamma}\Delta t=\infty, \quad t_1,t_2,t_3\in\mathbb{T}, \tag{26}$$

then every nonoscillatory solution of system (1) *with* $\lambda=-1$ *is of Type (b). In addition, if* (22) *holds, then system* (1) *is almost oscillatory.*

Proof. Suppose that (x,y,z) is a nonoscillatory solution of system (1) with $\lambda=-1$. Then by Lemma 2.6, (x,y,z) is of either Type (a) or Type (b). Suppose that (x,y,z) is a Type (a) solution. Without loss of generality, assume $x(t)>0$, $x(\tau(t))>0$, $y(t)>0$, $y(\tau(t))>0$, $z(t)>0$ for $t\geq t_0$, $t_0\in\mathbb{T}$. Integrating the second equation of system (1) from $t_1\geq t_0$, $t_1\in\mathbb{T}$ to $\tau(t)$ and using the positivity of y yield

$$y(\tau(t))\geq y(\tau(t))-y(t_1)=\int_{t_1}^{\tau(t)} b(s)g(z(\tau(s)))\Delta s, \quad \tau(t)\geq t_1.$$

By (3) and (4), there exist $G>0$ and $t_2\geq t_1$, $t_2\in\mathbb{T}$ such that

$$g(z(\tau(t)))\geq Gz^{\beta}(\tau(t)), \ \tau(t)\geq t_2.$$

Therefore, we obtain

$$\begin{aligned} y(\tau(t)) &\geq G\int_{t_1}^{\tau(t)} b(s)z^{\beta}(\tau(s))\Delta s \geq G\int_{t_1}^{\tau(t)} b(s)z^{\beta}(s)\Delta s \\ &\geq Gz^{\beta}(t)\int_{t_1}^{\tau(t)} b(s)\Delta s, \quad \tau(t)\geq t_2 \end{aligned}$$

or

$$y^{\alpha}(\tau(t)) \geq G^{\alpha} z^{\alpha\beta}(t) \left(\int_{t_1}^{\tau(t)} b(s)\Delta s \right)^{\alpha}, \quad \tau(t) \geq t_2. \tag{27}$$

By (3), (4) and (27), there exist $F > 0$ and $t_3 \geq t_2$, $t_3 \in \mathbb{T}$ such that

$$f(y(\tau(t))) \geq F y^{\alpha}(\tau(t)) \geq F G^{\alpha} z^{\alpha\beta}(t) \left(\int_{t_1}^{\tau(t)} b(s)\Delta s \right)^{\alpha}, \quad \tau(t) \geq t_3. \tag{28}$$

Integrating the first equation of system (1) from $t_3 \geq t_2$ to $\tau(t)$ and using (28)

$$\begin{aligned} x(\tau(t)) &\geq x(\tau(t)) - x(t_3) \\ &= \int_{t_3}^{\tau(t)} a(s) f(y(\tau(s)))\Delta s \\ &\geq FG^{\alpha} \int_{t_3}^{\tau(t)} a(s) z^{\alpha\beta}(s) \left(\int_{t_1}^{\tau(s)} b(v)\Delta v \right)^{\alpha} \Delta s \\ &\geq FG^{\alpha} z^{\alpha\beta}(t) \int_{t_3}^{\tau(t)} a(s) \left(\int_{t_1}^{\tau(s)} b(v)\Delta v \right)^{\alpha} \Delta s, \quad \tau(t) \geq t_3. \end{aligned}$$

or

$$x^{\gamma}(\tau(t)) > F^{\gamma} G^{\alpha\gamma} z^{\alpha\beta\gamma}(t) \left[\int_{t_3}^{\tau(t)} a(s) \left(\int_{t_1}^{\tau(s)} b(v)\Delta v \right)^{\alpha} \Delta s \right]^{\gamma}, \quad \tau(t) \geq t_3.$$

By (3) and (4), there exist $H > 0$ and $t_4 \geq t_3$ such that

$$h(x(\tau(t))) \geq H x^{\alpha}(\tau(t)), \quad \tau(t) \geq t_4.$$

From the third equation of system (1), we have

$$\begin{aligned} -z^{\Delta}(t) &= c(t) h(x(\tau(t))) \\ &\geq H c(t) x^{\gamma}(\tau(t)) \\ &> F^{\alpha} H G^{\alpha\gamma} c(t) z^{\alpha\beta\gamma}(t) \left[\int_{t_3}^{\tau(t)} a(s) \left(\int_{t_1}^{\tau(s)} b(v)\Delta v \right)^{\alpha} \Delta s \right]^{\gamma}, \quad \tau(t) \geq t_4. \end{aligned}$$

Dividing both sides of the above inequality by $z^{\alpha\beta\gamma}(t)$, we have

$$\frac{-z^{\Delta}(t)}{z^{\alpha\beta\gamma}(t)} > F^{\alpha} H G^{\alpha\gamma} c(t) \left[\int_{t_3}^{\tau(t)} a(s) \left(\int_{t_1}^{\tau(s)} b(v)\Delta v \right)^{\alpha} \Delta s \right]^{\gamma}, \quad \tau(t) \geq t_4.$$

Integrating the above inequality from t_4 to t yields

$$\int_{t_4}^{t} \frac{-z^{\Delta}(t)}{z^{\alpha\beta\gamma}(t)} \Delta t > F^{\alpha} H G^{\alpha\gamma} \int_{t_4}^{t} c(p) \left[\int_{t_3}^{\tau(t)} a(s) \left(\int_{t_1}^{\tau(s)} b(v)\Delta v \right)^{\alpha} \Delta s \right]^{\gamma} \Delta p, \quad t \geq t_4.$$

By [1], the left hand side of the above inequality is finite as $t \to \infty$, but this contradicts (26). Therefore, (x, y, z) can not be a Type (a) solution. So every nonoscillatory solution of system (1) is of Type (b). This implies that $\lim_{t\to\infty} x(t)$ is finite. Then by Lemma 2.7, we have $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} z(t) = 0$. By Theorem 3.2, $\lim_{t\to\infty} x(t) = 0$. So this completes the proof.

4 Almost Oscillatory System (1) when $\lambda = 1$

The last two results in this section are new for the discrete case.

Theorem 4.1 *Let conditions* (2), (3) *and* (14) *hold. Then system* (1) *with* $\lambda = 1$ *is almost oscillatory.*

Proof. It follows from Lemma 2.3 that nonoscillatory solutions of system (1) are either Type (a) or Type (c) solution of system (1). Assume that (x, y, z) is a Type (c) solutions of system (1). Without loss of generality, assume that there exists $t_0 \in \mathbb{T}$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $y(t) > 0$, $y(\tau(t)) > 0$, and $z(t) < 0$ for $t \geq t_0$. Since x is eventually increasing, there exist $t_1 \geq t_0$, $t_1 \in \mathbb{T}$ and $L > 0$ such that

$$h(x(\tau(t))) \geq L, \quad \tau(t) \geq t_1. \tag{29}$$

Integrating the third equation of system (1) from t_1 to t and using (29) we get

$$z(t) \geq z(t_1) + L \int_{t_1}^{t} c(s)\Delta s, \quad \tau(t) \geq t_1.$$

So (14) implies $\lim_{t\to\infty} z(t) = \infty$. This contradicts the assumptions on z. Therefore solutions of system (1) can not be of Type (c). If (x, y, z) is a Type (a) solution, then from Lemma 2.4 and equation (14), we obtain (5). This completes the proof.

For the next two theorems, we assume that

$$\int_{T}^{\infty} c(s)\Delta s < \infty, \quad T \in \mathbb{T}.$$

Theorem 4.2 *Let* (2) *and* (3) *hold. Assume that there exist positive constants* F, H *and* α, γ *such that*

$$\frac{f(u)}{\Phi_\alpha(u)} \geq F, \quad \frac{h(u)}{\Phi_\gamma(u)} \geq H \quad \textit{for large } u \neq 0, \tag{30}$$

and

$$\int_{t_3}^{\infty} c(r) \left(\int_{t_2}^{\tau(r)} a(s) \left(\int_{t_1}^{\tau(s)} b(\eta)\Delta\eta \right)^{\alpha} \Delta s \right)^{\gamma} \Delta r = \infty, \quad t_1, t_2, t_3 \in \mathbb{T}. \tag{31}$$

Then any Type (a) solution (x, y, z) *of system* (1) *with* $\lambda = 1$ *satisfies* (5).

Proof. Let (x, y, z) be a Type (a) solution of system (1) such that $x(\tau(t)) > 0$, $y(\tau(t)) > 0$, $z(\tau(t)) > 0$ for $t \geq t_0$. By (9), we have

$$y(t) \geq y(t_2) + M \int_{t_2}^{t} b(s)\Delta s \geq M \int_{t_2}^{t} b(s)\Delta s.$$

There exists $t_3 \in \mathbb{T}$, $t_3 \geq t_2$ such that

$$y(\tau(t)) \geq M \int_{t_3}^{\tau(t)} b(s)\Delta s, \quad \tau(t) \geq t_3,$$

and so

$$y^{\alpha}(\tau(t)) \geq M^{\alpha}\left(\int_{t_3}^{\tau(t)} b(s)\Delta s\right)^{\alpha}, \quad \tau(t) \geq t_3. \tag{32}$$

By (30), there exist $t_4 \in \mathbb{T}, t_4 \geq t_3$ and $F > 0$ such that

$$f(y(\tau(t))) \geq F y^{\alpha}(\tau(t)) \geq F M^{\alpha}\left(\int_{t_3}^{\tau(t)} b(s)\Delta s\right)^{\alpha}, \quad \tau(t) \geq t_4, \tag{33}$$

where we used (32). Integrating the first equation of system (1) from t_4 to t and using (33) yield

$$\begin{aligned} x(t) \geq x(t) - x(t_4) &= \int_{t_4}^{t} a(s) f(y(\tau(s)))\Delta s \\ &\geq F M^{\alpha} \int_{t_4}^{t} a(s)\left(\int_{t_3}^{\tau(s)} b(\eta)\Delta\eta\right)^{\alpha} \Delta s, \quad \tau(t) \geq t_4. \end{aligned}$$

Then there exists $t_5 \in \mathbb{T},\ t_5 \geq t_4$ such that

$$x(\tau(t)) \geq F M^{\alpha} \int_{t_4}^{\tau(t)} a(s)\left(\int_{t_3}^{\tau(s)} b(\eta)\Delta\eta\right)^{\alpha} \Delta s, \quad \tau(t) \geq t_5$$

or

$$x^{\gamma}(\tau(t)) \geq F^{\gamma} M^{\alpha\gamma}\left(\int_{t_4}^{\tau(t)} a(s)\left(\int_{t_3}^{\tau(s)} b(\eta)\Delta\eta\right)^{\alpha} \Delta s\right)^{\gamma}, \quad \tau(t) \geq t_5.$$

Using the third equation of system (1), (30) and the above inequality, we have

$$\begin{aligned} z^{\Delta}(t) &= c(t) h(x(\tau(t))) \\ &\geq c(t) H x^{\gamma}(\tau(t)) \\ &\geq F^{\gamma} M^{\alpha\gamma} c(t)\left(\int_{t_4}^{\tau(t)} a(s)\left(\int_{t_3}^{\tau(s)} b(\eta)\Delta\eta\right)^{\alpha} \Delta s\right)^{\gamma}, \quad \tau(t) \geq t_5. \end{aligned}$$

Integrating the above inequality from t_5 to t we get

$$\begin{aligned} z(t) &> z(t) - z(t_5) \\ &\geq F^{\gamma} M^{\alpha\gamma} \int_{t_5}^{t} c(s)\left(\int_{t_4}^{\tau(s)} a(\eta)\left(\int_{t_3}^{\tau(\eta)} b(r)\Delta r\right)^{\alpha} \Delta\eta\right)^{\gamma} \Delta s, \quad \tau(t) \geq t_5. \end{aligned}$$

So as $t \to \infty$, $z(t) \to \infty$ by (31). The proof is complete by Lemma 2.4.

Example 4.1 Let $\mathbb{T} = \mathbb{Z}$. Then we consider system (17) with $\lambda = 1$. Assume conditions (3) and (18) hold and there exist positive constants F, H, α, γ such that (30) holds. If

$$\sum_{r=1}^{\infty} c_r \left(\sum_{s=1}^{r-l-1} a_s \left(\sum_{n=1}^{s-l-1} b_n\right)^{\alpha}\right)^{\gamma} = \infty, \tag{34}$$

then any Type (a) solution (x, y, z) of system (17) with $\lambda = 1$ satisfies (5) by Theorem 4.2.

Theorem 4.3 *Let conditions* (2), (3) *hold and* $\beta \leq 1$. *Assume that there exist positive constants* G, β *such that*

$$\frac{g(u)}{\Phi_\beta(u)} \geq G \quad \textit{for large } u \neq 0, \tag{35}$$

where g *is an odd function. If*

$$\int_T^\infty c(s)\left(\int_T^{\sigma(s)} b(v)\Delta v\right)\Delta s = \infty, \quad T \in \mathbb{T}, \tag{36}$$

then every nonoscillatory solution of system (1) *with* $\lambda = 1$ *is a strongly monotone solution. In addition, if* (31) *holds, then system* (1) *with* $\lambda = 1$ *is almost oscillatory.*

Proof. Assume (x, y, z) is a Type (c) solution of system (1). Without loss of generality, assume that there exists $t_0 \in \mathbb{T}$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $y(t) > 0$, $y(\tau(t)) > 0$, $z(t) < 0$, $t \geq t_0$. Since x is eventually increasing, from (3) there exist $K > 0$ and $t_1 \geq t_0$, $t_1 \in \mathbb{T}$ such that $h(x(\tau(t))) \geq K$, $\tau(t) \geq t_1$. By Lemma 2.5, $\lim_{t\to\infty} z(t) = 0$. Then integrating the third equation of system (1) from t to ∞ yields

$$-z(t) = \int_t^\infty c(s)h(x(\tau(s)))\Delta s \geq K\int_t^\infty c(s)\Delta s.$$

From (35), there exist $t_2 \geq t_1$, $t_2 \in \mathbb{T}$ and $G > 0$ such that

$$g(-z(\tau(t))) \geq G(-z(\tau(t)))^\beta \geq G(-z(t)) \geq GK\int_t^\infty c(s)\Delta s, \quad \tau(t) \geq t_2. \tag{37}$$

Integrating the second equation of system (1) from t_2 to t, we have

$$y(t) - y(t_2) = \int_{t_2}^t b(s)g(z(\tau(s)))\Delta s$$

or

$$-y(t) + y(t_2) = \int_{t_2}^t b(s)g(-z(\tau(s)))\Delta s.$$

Using (37), we have

$$-y(t) + y(t_2) \geq GK\int_{t_2}^t b(s)\left(\int_s^\infty c(v)\Delta v\right)\Delta s, \quad \tau(t) \geq t_2. \tag{38}$$

Using Remark 1.1 for (38), we get

$$-y(t) + y(t_2) \geq GK\int_{t_2}^t c(s)\left(\int_{t_2}^{\sigma(s)} b(v)\Delta v\right)\Delta s, \quad \tau(t) \geq t_2.$$

As $t \to \infty$ and using (36), we get a contradiction with the boundedness of y. The second part follows from Theorem 4.2.

Example 4.2 Let $\mathbb{T} = \mathbb{Z}$. Then we consider system (17) with $\lambda = 1$. Assume conditions (3) and (18) hold and $\beta \leq 1$. There exist positive constants G, β such that (35) holds. If

$$\sum_{s=1}^{\infty} c_s \left(\sum_{r=1}^{s-l} b_r \right) = \infty, \tag{39}$$

then every nonoscillatory solution of system (17) with $\lambda = 1$ is a strongly monotone solution. In addition, if (34) holds, then system (17) with $\lambda = 1$ is almost oscillatory by Theorem 4.3.

5 Conclusion

In this paper, we consider oscillation and asymptotic behaviour of solutions of system (1) depending on $\lambda = \pm 1$. We conclude that system (1) with $\lambda = \pm 1$ is almost oscillatory, independently of the nonlineriatics, if (14) holds. However, if (20) holds, then system (1) is almost oscillatory depending on the sign of λ and the types of nonlinearities.

6 Acknowledgement

The first author is supported by Turkish Government and Dicle University.

References

[1] Akın-Bohner, E., Dŏslă, Z., and Lawrence, B. Oscillatory properties for three dimensional dynamic systems. *Nonlinear Anal.* **69** (2) (2008) 483-494.

[2] Akın-Bohner, E., Dŏslă, Z., and Lawrence, B. Almost oscillatory three-dimensional dynamical system. *Adv. Difference Equ.* **46** (2012) 14 pages.

[3] Bohner, M. and Peterson, A. *Dynamic Equations on Time Scales.* Birkhäuser Boston Inc., Boston, 2001.

[4] Bohner, M. and Peterson, A. *Advances in Dynamic Equations on Time Scales.* Birkhäuser Boston Inc., Boston, 2003.

[5] Kiguradze, I. T. and Chanturia, T. A. *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations.* Translated from the 1985 Russian original. Kluwer, Dordrecht, 1993.

[6] Schmeidel, E. Oscillation of nonlinear three-dimensional difference systems with delays. *Math. Bohem.* **135** (2) (2010) 163–170.

[7] Schmeidel, E. Boundedness of solutions of nonlinear three-dimensional difference systems with delays. *Fasc. Math.* **44** (2010) 107–113.

[8] Thandapani, E. and Ponnammal, B. Oscillatory properties of solutions of three-dimensional difference systems. *Math. Comput. Modelling* **45** (5-6) (2005) 641–650.