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Oscillation Criteria for Four-Dimensional Time-Scale Systems

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Abstract. In this paper, we obtain oscillation and nonoscillation criteria for solutions to four-dimensional systems of first-order dynamic equations on time scales. Especially, we are interested in the conditions which insure that every solution is oscillatory in the sub-linear, half-linear, and super-linear cases. Our approach is based on the sign of the components of nonoscillatory solutions. Several examples are included to highlight our main results.

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1. Introduction

We investigate four-dimensional dynamic systems of the form

$$\begin{cases} x^{\Delta}(t) = a(t)y^{\alpha}(t) \\ y^{\Delta}(t) = b(t)z^{\beta}(t) \\ z^{\Delta}(t) = c(t)w^{\gamma}(t) \\ w^{\Delta}(t) = -d(t)x^{\lambda}(\sigma(t)) \end{cases} \tag{1.1}$$

on a time scale $\mathbb{T}$, i.e., a closed subset of real numbers, where the coefficient functions $a, b, c, d \in C_{rd}\left([t_0,\infty)_{\mathbb{T}}, (0,\infty)\right)$ and $\alpha, \beta, \gamma, \lambda$ are the ratios of odd positive integers. Here, C_{rd} is the set of rd-continuous functions and $[t_0,\infty)_{\mathbb{T}} := [t_0,\infty) \cap \mathbb{T}$. Throughout this paper, we assume

$$\int_{t_0}^{\infty} a(t)\Delta t = \int_{t_0}^{\infty} b(t)\Delta t = \int_{t_0}^{\infty} c(t)\Delta t = \infty \tag{1.2}$$

and consider time scales unbounded. By a solution (x, y, z, w) of system (1.1), we mean that functions x, y, z, w are delta-differentiable, their first delta-derivatives are rd-continuous, and satisfy system (1.1) for all $t \geq t_0$. We call (x, y, z, w) a *proper* solution if it is defined on $[t_0,\infty)_{\mathbb{T}}$ and for $t \geq t_0$, $\sup\{|x(s)|, |y(s)|, |z(s)|, |w(s)| : s \in [t,\infty)_{\mathbb{T}}\} > 0$. A solution (x, y, z, w) of

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system (1.1) is said to be *oscillatory* if all of its components x, y, z, w are oscillatory, i.e., neither eventually positive nor eventually negative. Otherwise, it is said to be *nonoscillatory*. Obviously, if one component of a solution of system (1.1) is eventually of one sign, then all its components are eventually of one sign and so nonoscillatory solutions have all components nonoscillatory. We are also interested in system (1.1) in the sub-linear case, half-linear case, and super-linear case, that is, when $\alpha\beta\gamma\lambda < 1$, $\alpha\beta\gamma\lambda = 1$, and $\alpha\beta\gamma\lambda > 1$, respectively.

Motivated by [5,6], we establish some oscillation results for system (1.1) on time scales. In the next section, we present some auxiliary lemmas which are needed in the proof of our results and we consider two types of nonoscillatory solutions: one type when all components have the same sign and the other type when the third component has a different sign. In the following sections, we consider the properties of these types including the asymptotic behaviors. Our approach is based on the integral conditions of the coefficient functions a, b, c and d with the products of $\alpha, \beta, \gamma, \lambda$. We also illustrate the results by examples. Finally, we introduce the conditions which insure that every solution of system (1.1) is oscillatory in the sub-linear, half-linear and super-linear cases.

2. Preliminary Results

We only include preliminary results in this section. Nevertheless, we suggest readers the books by Bohner and Peterson [3,4] for an introduction to time-scale calculus.

The following lemma is essential to establish our main theorems for the sub-linear, half-linear and super-linear cases. Its proof follows from the chain rule on a time scale, see [1].

Lemma 2.1. *Let* $x \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$.

(i) *If* $0 < \eta < 1$ *and* $x^\Delta(t) < 0$ *on* $\mathbb{T}$, *then*

$$\int_T^\infty -\frac{x^\Delta(t)}{x^\eta(t)}\Delta t < \infty, \quad T \in \mathbb{T}.$$

(ii) *If* $\eta > 1$ *and* $x^\Delta(t) > 0$ *on* $\mathbb{T}$, *then*

$$\int_T^\infty \frac{x^\Delta(t)}{x^\eta(\sigma(t))}\Delta t < \infty, \quad T \in \mathbb{T}.$$

Using the sign of the components, one can observe the following: let (x, y, z, w) be a nonoscillatory solution of system (1.1). Without loss of generality, assume that $x(t) > 0$ for $t \geq t_0$, $t_0 \in \mathbb{T}$. From the fourth equation of system (1.1), w is strictly decreasing. Hence, it is of one sign. Continuing by the same argument, we get z and y are monotone and of one sign for large t too. The remaining cases when any of the components y, z, w are eventually positive or negative are proved similarly. Therefore, if one of the components of a solution (x, y, z, w) is eventually of one sign, then all of its components are eventually of one sign. In other words, any nonoscillatory solution of system (1.1) has all components nonoscillatory.

The next lemma shows that any nonoscillatory solution (x, y, z, w) of system (1.1) has two types when (1.2) holds.

Lemma 2.2. *Any nonoscillatory solution* (x, y, z, w) *of system* (1.1) *such that* $x(t) > 0$ *for large* $t \in \mathbb{T}$ *is one of the following types:*

$$\text{Type }(a): x > 0, y > 0, z > 0, w > 0 \text{ eventually}$$
$$\text{Type }(b): x > 0, y > 0, z < 0, w > 0 \text{ eventually.}$$

Proof. Let (x, y, z, w) be a nonoscillatory solution of system (1.1). Without loss of generality, assume that $x(t) > 0$ for $t \geq T$, $T \in \mathbb{T}$. Then we first assume that there exists a solution such that $y(t) > 0$, $z(t) < 0$, and $w(t) < 0$ for $t \geq T$. The negativity of w and the third equation of system (1.1) show that $z(t)$ is nonincreasing for $t \geq T$. Therefore, there exist $t_0 \geq T$, $t_0 \in \mathbb{T}$ and $k > 0$ such that $z(t) \leq -k$ for $t \geq t_0$. Plugging this inequality in the integration of the second equation from t_0 to t we get

$$y(t) - y(t_0) \leq -k^\beta \int_{t_0}^t b(s)\Delta s, \quad t \geq t_0.$$

Then $y(t) \to -\infty$ as $t \to \infty$, but this contradicts the fact that $y(t) > 0$ for large t. The case when $y(t) < 0$, $z(t) > 0$, and $w(t) > 0$ is similar and hence is eliminated. Now assume that there exists a nonoscillatory solution of system (1.1) such that $y(t) < 0$, $z(t) < 0$ for $t \geq T$. The negativity of z and the second equation of system (1.1) yield $y(t)$ is nonincreasing for $t \geq T$. Hence, there exist $t_0 \geq T$, $t_0 \in \mathbb{T}$ and $l > 0$ such that $y(t) \leq -l$ for $t \geq t_0$. Plugging this inequality in the integration of the first equation from t_0 to t yields

$$x(t) - x(t_0) \leq -l^\alpha \int_{t_0}^t a(s)\Delta s, \quad t \geq t_0.$$

Then $x(t) \to -\infty$ as $t \to \infty$, but this contradicts the fact that $x(t) > 0$ for large t. Next, assume that there exists a nonoscillatory solution of system (1.1) such that $z(t) > 0$, $w(t) < 0$ for $t \geq T$. The positivity of x and the fourth equation of system (1.1) yield $w(t)$ which is nonincreasing for $t \geq T$. Hence, there exist $t_0 \geq T$, $t_0 \in \mathbb{T}$ and $m > 0$ such that $w(t) \leq -m$ for $t \geq t_0$. Using this inequality and integrating the third equation from t_0 to t, we get

$$z(t) - z(t_0) \leq -m^\gamma \int_{t_0}^t c(s)\Delta s, \quad t \geq t_0.$$

Then $z(t) \to -\infty$ as $t \to \infty$, but this contradicts the assumption $z(t) > 0$ for large t. □

To show that system (1.1) is oscillatory, we first try the divergence of the single integral of d.

Lemma 2.3. *System* (1.1) *is oscillatory if*

$$\int_{t_0}^\infty d(t)\Delta t = \infty. \tag{2.1}$$

Proof. By Lemma 2.2, any nonoscillatory solution of system (1.1) is either Type (a) or Type (b). Let (x, y, z, w) be of a Type (a) solution of system (1.1) such that $x(t) > 0$ for $t \geq T$. The positivity of y and the first equation of system (1.1) show that $x(t)$ is nondecreasing for $t \geq T$. Therefore, there exist $t_0 \geq T$, $t_0 \in \mathbb{T}$ and $k > 0$ such that $x(t) \geq k$ for $t \geq t_0$. Then using this inequality and the integration of the fourth equation from t_0 to t give us

$$w(t) \leq -k^{\lambda} \int_{t_0}^{t} d(s)\Delta s, \quad t \geq t_0.$$

As $t \to \infty$, $w(t) \to -\infty$ by (2.1). But, this is a contradiction because of the assumption $w(t) > 0$ for large t. The discussion above is also valid for Type (b) solutions because the sign of z is not needed in this proof. Therefore, system (1.1) does not have any nonoscillatory solutions and so the proof is completed. □

Now as a result of the discussion above, from now on we will assume that

$$\int_{t_0}^{\infty} d(t)\Delta t < \infty. \tag{2.2}$$

3. Type (a) Solutions

In this section, we investigate not only nonoscillatory criteria, but also the asymptotic behavior of Type (a) solutions. The following property of Type (a) solutions in the discrete case can be found in [7].

Proposition 3.1. *Every solution (x, y, z, w) of Type (a) of system (1.1) satisfies*

$$I \left(\int_{t}^{\infty} d(s)\Delta s \right)^{\gamma\beta\alpha} \leq x^{1-\lambda\gamma\beta\alpha}(\sigma(t)), \tag{3.1}$$

where $t \in \mathbb{T}$ is sufficiently large and

$$I = \int_{t_0}^{t} a(s) \left(\int_{t_0}^{s} b(r) \left(\int_{t_0}^{r} c(\eta)\Delta\eta \right)^{\beta} \Delta r \right)^{\alpha} \Delta s. \tag{3.2}$$

Proof. Let (x, y, z, w) be of a Type (a) solution of system (1.1) such that $x(t) > 0$ for $t \geq T$. Then integrating the third equation from t_0 to t yields

$$z(t) \geq \int_{t_0}^{t} c(s)w^{\gamma}(s)\Delta s, \quad t \geq t_0. \tag{3.3}$$

Since $w(t)$ is nonincreasing for $t \geq T$, (3.3) yields

$$z^{\beta}(t) \geq w^{\gamma\beta}(t) \left(\int_{t_0}^{t} c(s)\Delta s \right)^{\beta}, \quad t \geq t_0.$$

Now integrating the second equation of system (1.1) from t_0 to t and plugging the above inequality into the resulting inequality yield

$$y^{\alpha}(t) \geq w^{\gamma\beta\alpha}(t) \left(\int_{t_0}^{t} b(s) \left(\int_{t_0}^{s} c(r)\Delta r \right)^{\beta} \Delta s \right)^{\alpha}, \quad t \geq t_0, \tag{3.4}$$

where we use the monotonicity of w. Integrating the first equation of system (1.1) from t_0 to t and substituting (3.4) in the resulting integration give us

$$x(\sigma(t)) \geq w^{\gamma\beta\alpha}(t)I, \tag{3.5}$$

where we use the monotonicities of x and w, and I is defined as in (3.2). Integrating the fourth equation of system (1.1) from t to ∞ and using the monotonicity of x yield

$$w(t) \geq \int_t^\infty d(s)x^\lambda(\sigma(s))\Delta s \geq x^\lambda(\sigma(t)) \int_t^\infty d(s)\Delta s \tag{3.6}$$

and this implies

$$w^{\gamma\beta\alpha}(t) \geq x^{\lambda\gamma\beta\alpha}(\sigma(t)) \left(\int_t^\infty d(s)\Delta s\right)^{\gamma\beta\alpha}. \tag{3.7}$$

Therefore, from (3.5) and (3.7) we have

$$x(\sigma(t)) \geq Ix^{\lambda\gamma\beta\alpha}(\sigma(t)) \left(\int_t^\infty d(s)\Delta s\right)^{\gamma\beta\alpha}$$

which proves the desired result (3.1). □

Theorem 3.1. *Every nonoscillatory solution of system* (1.1) *is of Type* (a) *if any of the following conditions holds:*

(i) $\displaystyle\int_{t_0}^\infty c(t)\left(\int_t^\infty d(s)\Delta s\right)^\gamma \Delta t = \infty;$

(ii) $\displaystyle\int_{t_0}^\infty b(t)\left(\int_t^\infty c(r)\left(\int_r^\infty d(\tau)\Delta\tau\right)^\gamma \Delta r\right)^\beta \Delta s = \infty;$

(iii) $\alpha\beta\gamma\lambda < 1$, *and*
$$\int_{t_0}^\infty b(t)\left(\int_{t_0}^t a(s)\Delta s\right)^{\lambda\gamma\beta}\left(\int_t^\infty c(r)\left(\int_r^\infty d(\tau)\Delta\tau\right)^\gamma \Delta r\right)^\beta \Delta t = \infty;$$

(iv) $\lambda\gamma\beta\alpha > 1$, *and*
$$\int_{t_0}^\infty a(t)\left(\int_{\sigma(t)}^\infty b(s)\left(\int_s^\infty c(r)\left(\int_r^\infty d(\eta)\Delta\eta\right)^\gamma \Delta r\right)^\beta \Delta s\right)^\alpha \Delta t = \infty.$$

Proof. Since (1.2) holds, every nonoscillatory solution of system (1.1) is of either Type (a) or Type (b) by Lemma 2.2. Assume that (x, y, z, w) is of a Type (b) solution of system (1.1) such that $x(t) > 0$ for $t \geq T$.

Assume (i) holds. Since the monotonicities and the signs of x and w are same for both types, (3.6) holds not only for Type (a) solutions but also for Type (b) solutions of system (1.1). Substituting (3.6) in the integration of the third equation from t_0 to t yields

$$-z(t_0) \geq x^{\lambda\gamma}(t_0)\int_{t_0}^t c(s)\left(\int_s^\infty d(r)\Delta r\right)^\gamma \Delta s, \quad t \geq t_0 \tag{3.8}$$

following from the monotonicity of x. As $t \to \infty$, the right-hand side of (3.8) approaches to ∞ by (i), but then this contradicts the boundedness of z. Therefore, (x, y, z, w) is of Type (a) solution.

Assume that (ii) holds. Since w is positive, from the third equation of system (1.1) we have that $z(t)$ is nondecreasing for $t \geq T$. Therefore, by integrating the third equation of system (1.1) from t to ∞ and using the inequality (3.6), we have

$$-z(t) \geq x^{\lambda\gamma}(t) \int_t^\infty c(s) \left(\int_s^\infty d(r)\Delta r \right)^\gamma \Delta s, \quad t \geq t_0, \tag{3.9}$$

where we use the monotonicity of x. The negativity of z and the second equation of system (1.1) give us that $y(t)$ is nonincreasing for $t \geq T$. Therefore, integrating the second equation from t_0 to t and plugging (3.9) into the resulting integration yield

$$y(t_0) \geq x^{\lambda\gamma\beta}(t_0) \int_{t_0}^t b(s) \left(\int_s^\infty c(r) \left(\int_r^\infty d(\tau)\Delta\tau \right)^\gamma \Delta r \right)^\beta \Delta s. \tag{3.10}$$

As $t \to \infty$, the right-hand side of the inequality (3.10) approaches to ∞ by (ii). On the other hand, this contradicts the boundedness of y. Hence, we have shown that (x, y, z, w) is of Type (a) solution.

Assume that (iii) holds. By integrating the first equation of system (1.1) from t_0 to t and using the monotonicity of y, we get

$$x(t) \geq \int_{t_0}^t a(s) y^\alpha(s) \Delta s \tag{3.11}$$

$$\geq y^\alpha(t) \int_{t_0}^t a(s)\Delta s, \quad t \geq t_0. \tag{3.12}$$

Substituting (3.9) in the second equation of system (1.1) yields for $t \geq t_0$

$$-y^\Delta(t) = b(t)(-z^\beta(t)) \geq x^{\lambda\gamma\beta}(t) b(t) \left(\int_t^\infty c(s) \left(\int_s^\infty d(r)\Delta r \right)^\gamma \Delta s \right)^\beta.$$

Finally, substituting (3.12) in the above inequality gives us

$$-y^\Delta(t) \geq b(t) y^{\lambda\gamma\beta\alpha}(t) \left(\int_{t_0}^t a(s)\Delta s \right)^{\lambda\gamma\beta} \left(\int_t^\infty c(s) \left(\int_s^\infty d(r)\Delta r \right)^\gamma \Delta s \right)^\beta.$$

Dividing both sides of the inequality above by $y^{\lambda\gamma\beta\alpha}$ and integrating both sides of the resulting inequality from t_0 to t yield

$$\int_{t_0}^t -\frac{y^\Delta(s)}{y^{\lambda\gamma\beta\alpha}(s)} \Delta s$$
$$\geq \int_{t_0}^t b(s) \left(\int_{t_0}^s a(\eta)\Delta\eta \right)^{\lambda\gamma\beta} \left(\int_s^\infty c(r) \left(\int_r^\infty d(\tau)\Delta\tau \right)^\gamma \Delta r \right)^\beta \Delta s.$$

As $t \to \infty$, $\int_{t_0}^\infty -\frac{y^\Delta(s)}{y^{\lambda\gamma\beta\alpha}(s)} \Delta s = \infty$ by (iii). However, $\int_{t_0}^\infty -\frac{y^\Delta(s)}{y^{\lambda\gamma\beta\alpha}(s)} \Delta s < \infty$ by Lemma 2.1 (i) so this gives a contradiction and completes the proof. Therefore, (x, y, z, w) is of Type (a) solution.

Assume that (iv) holds. Integrating the second equation of system (1.1) from $\sigma(t)$ to ∞ and the monotonicity of y yield

$$y(t) \geq y(\sigma(t)) \geq \int_{\sigma(t)}^{\infty} b(s)(-z^{\beta}(s))\Delta s. \tag{3.13}$$

Substituting (3.9) and (3.13) gives

$$y(t) \geq x^{\lambda\gamma\beta}(\sigma(t)) \int_{\sigma(t)}^{\infty} b(s) \left(\int_{s}^{\infty} c(r) \left(\int_{r}^{\infty} d(\tau)\Delta\tau \right)^{\gamma} \Delta r \right)^{\beta} \Delta s, \tag{3.14}$$

where we use the monotonicity of x. Now after plugging (3.14) into the first equation system (1.1), dividing both sides of the resulting inequality by $x^{\lambda\gamma\beta\alpha}(\sigma(t))$ and then integrating from t_0 to t, we obtain

$$\begin{aligned} &\int_{t_0}^{t} \frac{x^{\Delta}(s)}{x^{\lambda\gamma\beta\alpha}(\sigma(s))}\Delta s \\ &\quad \geq \int_{t_0}^{t} a(s) \left(\int_{\sigma(s)}^{\infty} b(r) \left(\int_{r}^{\infty} c(\tau) \left(\int_{\tau}^{\infty} d(\eta)\Delta\eta \right)^{\gamma} \Delta\tau \right)^{\beta} \Delta r \right)^{\alpha} \Delta s. \end{aligned}$$

As $t \to \infty$, $\int_{t_0}^{\infty} \frac{x^{\Delta}(s)}{x^{\lambda\gamma\beta\alpha}(s)}\Delta s = \infty$ by (iv). However, $\int_{t_0}^{\infty} \frac{x^{\Delta}(t)}{x^{\lambda\gamma\beta\alpha}(\sigma(t))}\Delta t < \infty$ by Lemma 2.1 (ii). So this gives a contradiction and shows that (x, y, z, w) has to be of Type (a) solution of system (1.1). □

Since $\int_{t_0}^{\infty} d(t)\Delta t < \infty$, we have

$$\int_{t_0}^{\infty} c(t) \left(\int_{t}^{\infty} d(s)\Delta s \right) \Delta t = \int_{t_0}^{\infty} d(t) \left(\int_{t_0}^{\sigma(t)} c(s)\Delta s \right) \Delta t,$$

see [2]. Therefore, in the special case of $\gamma = 1$ in part (i) of Theorem 3.1, we get the following nonoscillation criteria.

Remark 3.1. If $\int_{t_0}^{\infty} d(t) \left(\int_{t_0}^{\sigma(t)} c(s)\Delta s \right) \Delta t = \infty$, then every nonoscillatory solution of system (1.1) is of Type (a).

Finding an integral condition for Type (a) solutions when $\lambda\gamma\beta\alpha = 1$ is still open for discussion. Nevertheless, we have the following corollary.

Corollary 3.1. *Every nonoscillatory solution of system* (1.1) *is of Type* (a) *if* $\lambda\gamma\beta\alpha = 1$ *and*

$$\limsup_{t\to\infty} \left(\int_{\sigma(t)}^{\infty} b(s) \left(\int_{s}^{\infty} c(r) \left(\int_{r}^{\infty} d(\eta)\Delta\eta \right)^{\gamma} \Delta r \right)^{\beta} \Delta s \right)^{\alpha} \left(\int_{t_0}^{t} a(s)\Delta s \right) > 1. \tag{3.15}$$

Proof. Let $\alpha\beta\gamma\lambda = 1$. Since (1.2) holds, every nonoscillatory solution of system (1.1) is of either Type (a) or Type (b) by Lemma 2.2. Assume (3.15) holds and (x, y, z, w) is of a Type (b) solution of system (1.1) such that $x(t) > 0$ for $t \geq T$. Then (3.12) and (3.14) hold. Plugging (3.14) into (3.12) yields

$$x(t) \geq x^{\lambda\gamma\beta\alpha}(\sigma(t)) \left(\int_{\sigma(t)}^{\infty} b(s) \left(\int_{s}^{\infty} c(r) \left(\int_{r}^{\infty} d(\tau)\Delta\tau \right)^{\gamma} \Delta r \right)^{\beta} \Delta s \right)^{\alpha} \int_{t_0}^{t} a(s)\Delta s. \tag{3.16}$$

Hence, after dividing the inequality above by $x^{\lambda\gamma\beta\alpha}(\sigma(t))$ and taking the lim sup of the resulting inequality, we get

$$\limsup_{t\to\infty} \left(\int_{\sigma(t)}^{\infty} b(s) \left(\int_{s}^{\infty} c(r) \left(\int_{r}^{\infty} d(\eta)\Delta\eta \right)^{\gamma} \Delta r \right)^{\beta} \Delta s \right)^{\alpha} \left(\int_{t_0}^{t} a(s)\Delta s \right) \leq 1$$

which contradicts (3.15). Therefore, (x, y, z, w) is of a Type (a) solution of (1.1). □

Remark 3.2. Any Type (a) solution (x, y, z, w) of system (1.1) satisfies the following:

(i) $\lim_{t\to\infty} x(t) = \infty$;
(ii) $\lim_{t\to\infty} y(t) = \infty$.

Proof. Let (x, y, z, w) be of a Type (a) solution of system (1.1) such that $x(t) > 0$ for $t \geq T$. Since z is positive, from the second equation of system (1.1) we have that $y(t)$ is nondecreasing for $t \geq T$. Hence, there exist $t_0 \geq T$, $t_0 \in \mathbb{T}$ and $k > 0$ such that $y(t) \geq k$ for $t \geq t_0$. Then (3.11) holds. This implies that

$$x(t) \geq k^{\alpha} \int_{t_0}^{t} a(s)\Delta s, \quad t \geq t_0. \tag{3.17}$$

As $t \to \infty$, we get $\lim_{t\to\infty} x(t) = \infty$.

Now since w is positive, from the third equation of system (1.1) we have that $z(t)$ is nondecreasing for $t \geq T$. Hence, there exist $t_0 \geq T$, $t_0 \in \mathbb{T}$ and $k > 0$ such that $z(t) \geq k$ for $t \geq t_0$. Integrating the second equation from t_0 to t and using this inequality give us

$$y(t) \geq \int_{t_0}^{t} b(s) z^{\beta}(s)\Delta s \tag{3.18}$$

$$\geq k^{\beta} \int_{t_0}^{t} b(s)\Delta s, \quad t \geq t_0. \tag{3.19}$$

As $t \to \infty$, we have $\lim_{t\to\infty} y(t) = \infty$. □

Let us consider the following example to illustrate Theorem 3.1.

Example 3.1. Let $\mathbb{T} = \mathbb{Z}$ and $t_0 = 1$. Consider the system

$$\begin{cases} \Delta x_n = \frac{19.3^n}{2^{n+3}} y_n \\ \Delta y_n = \frac{5.3^{\frac{9n}{5}}}{2^{2n+2}} z_n^{\frac{1}{5}} \\ \Delta z_n = \frac{2^{n+1}}{3^{1-n}} w_n \\ \Delta w_n = -\frac{1}{3^n} x_{n+1}^{\frac{1}{3}} \end{cases}. \tag{3.20}$$

Then $\int_1^\infty a(t)\Delta t = \lim_{T\to\infty}\sum_{n=1}^{T-1}\frac{19.3^n}{2^{n+3}} = \sum_{n=1}^{\infty}\frac{19.3^n}{2^{n+3}} = \frac{19}{8}\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n = \infty$ by geometric series. Similarly, $\int_1^\infty b(t)\Delta t = \int_1^\infty c(t)\Delta t = \infty$, and $\int_1^\infty d(t)\Delta t < \infty$. Furthermore,

$$\begin{aligned}\int_1^\infty c(t)\left(\int_t^\infty d(s)\Delta s\right)\Delta t &= \lim_{T\to\infty}\sum_{n=1}^{T-1} c_n\left(\sum_{k=n}^{\infty} d_n\right) = \sum_{n=1}^{\infty}\frac{2^{n+1}}{3^{1-n}}\left(\sum_{k=n}^{\infty}\frac{1}{3^k}\right)\\ &= \frac{2}{3}\sum_{n=1}^{\infty}6^n\left(\sum_{k=n}^{\infty}\frac{1}{3^k}\right) = \frac{2}{3}\sum_{n=1}^{\infty}6^n\frac{1}{3^n}\frac{3}{2} = \sum_{n=1}^{\infty}2^n\\ &= \infty.\end{aligned}$$

Therefore, every nonoscillatory solution of system (3.20) is of Type (a) by Theorem 3.1 (i). In fact, one can also show that $\left(\left(\frac{3}{2}\right)^{3n}, \left(\frac{3}{2}\right)^{2n}, 3^n, \frac{3}{2^n}\right)$ is of a Type (a) solution of (3.20).

4. Type (b) Solutions

The following property of Type (b) solutions in the discrete case is shown by Došlá and Krejčová in [7] and its proof follows from (3.16) immediately.

Proposition 4.1. *Every solution (x, y, z, w) of Type (b) of system (1.1) satisfies*

$$J^\alpha\int_{t_0}^{t} a(s)\Delta s \le \frac{x(t)}{x^{\lambda\gamma\beta\alpha}(\sigma(t))},$$

where $t\in\mathbb{T}$ is sufficiently large and

$$J = \int_{\sigma(t)}^{\infty} b(s)\left(\int_s^\infty c(r)\left(\int_r^\infty d(\eta)\Delta\eta\right)^\gamma\Delta r\right)^\beta\Delta s.$$

Theorem 4.1. *Every nonoscillatory solution of system (1.1) is of a Type (b) solution if any of the following conditions holds:*

(i) $\int_{t_0}^{\infty} d(t)\left(\int_{t_0}^{t} a(s)\Delta s\right)^\lambda\Delta t = \infty$;

(ii) $\int_{t_0}^{\infty} d(t)\left(\int_{t_0}^{t} a(s)\left(\int_{t_0}^{s} b(r)\Delta r\right)^\alpha\Delta s\right)^\lambda\Delta t = \infty$;

(iii) $\alpha\beta\gamma\lambda < 1$ *and*
$$\int_{t_0}^{\infty} d(t)\left(\int_{t_0}^{t} a(s)\left(\int_{t_0}^{s} b(r)\left(\int_{t_0}^{r} c(\tau)\Delta\tau\right)^\beta\Delta r\right)^\alpha\Delta s\right)^\lambda\Delta t = \infty;$$

(iv) $\alpha\beta\gamma\lambda = 1$ *and* $0<\varepsilon<1$
$$\int_{t_0}^{\infty} d(t)\left(\int_{t_0}^{t} a(s)\left(\int_{t_0}^{s} b(r)\left(\int_{t_0}^{r} c(\tau)\Delta\tau\right)^\beta\Delta r\right)^\alpha\Delta s\right)^{\lambda(1-\varepsilon)}\Delta t = \infty;$$

(v) $\alpha\beta\gamma\lambda > 1$ *and*
$$\int_{t_0}^{\infty} a(t) \left(\int_{t_0}^{t} b(s) \left(\int_{t_0}^{s} c(\tau)\Delta\tau \right)^{\beta} \Delta s \right)^{\alpha} \left(\int_{\sigma(t)}^{\infty} d(\eta)\Delta\eta \right)^{\gamma\beta\alpha} \Delta t = \infty.$$

Proof. Since (1.2) holds, every nonoscillatory solution of system (1.1) is of either Type (a) or Type (b) by Lemma 2.2. Assume that (x, y, z, w) is a Type (a) solution of system (1.1) such that $x(t) > 0$ for $t \geq T$.

Assume that (i) holds. Then (3.17) holds. Now integrating the fourth equation of system (1.1) from t_0 to t and plugging (3.17) into the resulting integration yield for $t \geq t_0$,

$$w(t) - w(t_0) = -\int_{t_0}^{t} d(s) x^{\lambda}(\sigma(s))\Delta s \leq -k^{\alpha} \int_{t_0}^{t} d(s) \left(\int_{t_0}^{s} a(r)\Delta r \right)^{\lambda} \Delta s$$

following from the monotonicity of x. Then as $t \to \infty$, $w(t) \to -\infty$ by (i). But this contradicts the boundedness of w. Therefore, (x, y, z, w) is of a Type (b) solution of system (1.1).

Assume that (ii) holds. After integrating the first equation from t_0 to t and using (3.19), we obtain

$$x^{\lambda}(\sigma(t)) \geq k^{\alpha\beta\lambda} \left(\int_{t_0}^{t} a(s) \left(\int_{t_0}^{s} b(r)\Delta r \right)^{\alpha} \Delta s \right)^{\lambda}, \quad t \geq t_0. \tag{4.1}$$

Integrating the fourth equation of system (1.1) from t_0 to t and plugging (4.1) into it, one can get

$$w(t) - w(t_0) \leq -k^{\alpha\beta\lambda} \int_{t_0}^{t} d(s) \left(\int_{t_0}^{s} a(r) \left(\int_{t_0}^{r} b(\tau)\Delta\tau \right)^{\alpha} \Delta r \right)^{\lambda} \Delta s, \quad t \geq t_0. \tag{4.2}$$

As $t \to \infty$, the right-hand side of (4.2) approaches to $-\infty$ by (ii). Therefore, $w(t) \to -\infty$. However, this contradicts the boundedness of w and completes the proof. Hence, (x, y, z, w) is of a Type (b) solution of system (1.1).

Assume that (iii) holds. Taking the λ power of (3.5) and then multiplying both sides of the inequality by $-d$ give us the right-hand side of the inequality of (3.5) being w^{Δ}, as follows:

$$w^{\Delta}(t) \leq -w^{\gamma\beta\alpha\lambda}(t) d(t) \left(\int_{t_0}^{t} a(s) \left(\int_{t_0}^{s} b(r) \left(\int_{t_0}^{r} c(\eta)\Delta\eta \right)^{\beta} \Delta r \right)^{\alpha} \Delta s \right)^{\lambda}.$$

Now dividing both sides of this inequality by $-w^{\gamma\beta\alpha\lambda}$ and integrating both sides of the resulting inequality from t_0 to t yield

$$\begin{aligned} &\int_{t_0}^{t} -\frac{w^{\Delta}(s)}{w^{\gamma\beta\alpha\lambda}(s)} \Delta s \\ &\quad \geq \int_{t_0}^{t} d(s) \left(\int_{t_0}^{s} a(r) \left(\int_{t_0}^{r} b(\tau) \left(\int_{t_0}^{\tau} c(\eta)\Delta\eta \right)^{\beta} \Delta\tau \right)^{\alpha} \Delta r \right)^{\lambda} \Delta s. \end{aligned}$$

As $t \to \infty$, $\int_{t_0}^{\infty} -\frac{w^{\Delta}(s)}{w^{\gamma\beta\alpha\lambda}(s)}\Delta s = \infty$ by (iii). However, $\int_{t_0}^{\infty} -\frac{w^{\Delta}(s)}{w^{\gamma\beta\alpha\lambda}(s)}\Delta s < \infty$ by Lemma 2.1 (i). So this gives a contradiction and hence (x, y, z, w) is of a Type (b) solution of system (1.1).

Assume that (iv) holds. Taking the $\lambda(1-\epsilon)$ power of both sides of (3.5) implies that

$$x^{\lambda(1-\epsilon)}(\sigma(t)) \geq w^{1-\epsilon}(t)\left(\int_{t_0}^{t} a(s)\left(\int_{t_0}^{s} b(r)\left(\int_{t_0}^{r} c(\eta)\Delta\eta\right)^{\beta}\Delta r\right)^{\alpha}\Delta s\right)^{\lambda(1-\epsilon)}. \tag{4.3}$$

Since x is nondecreasing, there exists $k > 0$ such that $x^{\lambda}(\sigma(t)) \geq k$ for large t. This yields

$$x^{\lambda(1-\epsilon)}(\sigma(t)) \leq \frac{x^{\lambda}(\sigma(t))}{k^{\epsilon}} \quad \text{for large } t.$$

Now using the above inequality together with (4.3) and multiplying both sides of the resulting inequality by d give us

$$-w^{\Delta}(t) \geq k^{\epsilon} w^{1-\epsilon}(t) d(t)\left(\int_{t_0}^{t} a(s)\left(\int_{t_0}^{s} b(r)\left(\int_{t_0}^{r} a(\eta)\Delta\eta\right)^{\beta}\Delta r\right)^{\alpha}\Delta s\right)^{\lambda(1-\epsilon)}.$$

Dividing this inequality by $w^{1-\epsilon}$ and integrating both sides of the resulting inequality from t_0 to t yield

$$\begin{aligned}&\int_{t_0}^{t} -\frac{w^{\Delta}(s)}{w^{1-\epsilon}(s)}\Delta s \\ &\quad\geq k^{\epsilon}\int_{t_0}^{t} d(s)\left(\int_{t_0}^{s} a(r)\left(\int_{t_0}^{r} b(\tau)\left(\int_{t_0}^{\tau} c(\eta)\Delta\eta\right)^{\beta}\Delta\tau\right)^{\alpha}\Delta r\right)^{\lambda(1-\epsilon)}\Delta s.\end{aligned}$$

As $t \to \infty$, $\int_{t_0}^{\infty} -\frac{w^{\Delta}(s)}{w^{1-\epsilon}(s)}\Delta s = \infty$ by (iv). However, by Lemma 2.1 (i) we obtain $\int_{t_0}^{\infty} -\frac{w^{\Delta}(s)}{w^{1-\epsilon}(s)}\Delta s < \infty$, $0 < \varepsilon < 1$. This gives a contradiction and hence (x, y, z, w) is of a Type (b) solution of system (1.1).

Assume (v) holds. Integrating the fourth equation of system (1.1) from $\sigma(t)$ to ∞ and using the monotonicity of x give us

$$w(\sigma(t)) \geq x^{\lambda}(\sigma(t))\int_{\sigma(t)}^{\infty} d(s)\Delta s. \tag{4.4}$$

After substituting (3.18) in the first equation of system (1.1) and then substituting (3.3) in the resulting inequality, we get

$$\begin{aligned}x^{\Delta}(t) &\geq a(t)\left(\int_{t_0}^{t} b(s) z^{\beta}(s)\Delta s\right)^{\alpha} \\ &\geq a(t)\left(\int_{t_0}^{t} b(s)\left(\int_{t_0}^{s} c(r) w^{\gamma}(r)\Delta r\right)^{\beta}\Delta s\right)^{\alpha}, \quad t \geq t_0.\end{aligned}$$

From the monotonicity of w, this inequality becomes

$$x^{\Delta}(t) \geq w^{\gamma\beta\alpha}(\sigma(t))a(t)\left(\int_{t_0}^{t} b(s)\left(\int_{t_0}^{s} c(r)\Delta r\right)^{\beta}\Delta s\right)^{\alpha}, \quad t \geq t_0. \tag{4.5}$$

Now plugging (4.4) into (4.5), i.e.,

$$\begin{aligned} &x^{\Delta}(t)\\ &\quad \geq x^{\lambda\gamma\beta\alpha}(\sigma(t))a(t)\left(\int_{t_0}^{t} b(s)\left(\int_{t_0}^{s} c(r)\Delta r\right)^{\beta}\Delta s\right)^{\alpha}\left(\int_{\sigma(t)}^{\infty} d(s)\Delta s\right)^{\gamma\beta\alpha}\Delta t, \end{aligned}$$

then dividing both sides of the above inequality by $x^{\lambda\gamma\beta\alpha}(\sigma(t))$, and integrating the resulting inequality from t_0 to t yield

$$\begin{aligned} &\int_{t_0}^{t} \frac{x^{\Delta}(s)}{x^{\lambda\gamma\beta\alpha}(\sigma(s))}\Delta s\\ &\quad \geq \int_{t_0}^{t} a(s)\left(\int_{t_0}^{s} b(r)\left(\int_{t_0}^{r} c(\tau)\Delta\tau\right)^{\beta}\Delta r\right)^{\alpha}\left(\int_{\sigma(s)}^{\infty} d(\eta)\Delta\eta\right)^{\gamma\beta\alpha}\Delta s. \end{aligned}$$

As $t \to \infty$, we get $\displaystyle\int_{t_0}^{\infty} \frac{x^{\Delta}(t)}{x^{\lambda\gamma\beta\alpha}(\sigma(t))}\Delta t = \infty$ by (v). However, it contradicts $\displaystyle\int_{t_0}^{\infty} \frac{x^{\Delta}(t)}{x^{\lambda\gamma\beta\alpha}(\sigma(t))}\Delta t < \infty$ by Lemma 2.1 (ii). Therefore, (x, y, z, w) is of a Type (b) solution of system (1.1). □

From changing the order of integration in part (i) of Theorem 4.1 when $\lambda = 1$, we obtain

$$\int_{t_0}^{t} d(s)\left(\int_{t_0}^{s} a(r)\Delta r\right)\Delta s = \int_{t_0}^{t} a(s)\left(\int_{\sigma(s)}^{t} d(r)\Delta r\right)\Delta t,$$

see [2]. Therefore, we have the following result.

Remark 4.1. If $\displaystyle\int_{t_0}^{\infty} a(s)\left(\int_{\sigma(s)}^{\infty} d(r)\Delta r\right)\Delta s = \infty$, then every nonoscillatory solution of system (1.1) is of Type (b).

Remark 4.2. Any Type (b) solution (x, y, z, w) of (1) satisfies $\lim_{t\to\infty} z(t) = 0$.

Proof. Let (x, y, z, w) be of a Type (b) solution of system (1.1) such that $x(t) > 0$ for large $t \in \mathbb{T}$. Then z is eventually negative increasing. Therefore, $\lim_{t\to\infty} z(t) = l \leq 0$. Suppose that $l < 0$, then from the monotonicity of z, we have $z(t) \leq l$ for large t. Integrating the second of system (1.1) from t_0 to t yields

$$y(t) - y(t_0) \leq l^{\beta}\int_{t_0}^{t} b(s)\Delta s, \quad t \geq t_0.$$

Letting $t \to \infty$ implies $\lim_{t\to\infty} y(t) = -\infty$. But, this contradicts the positivity of y. Hence, $\lim_{t\to\infty} z(t) = 0$. □

Corollary 4.1. *Every nonoscillatory solution of system* (1.1) *is of Type* (b) *if* $\alpha\beta\gamma\lambda = 1$ *and*

$$\limsup_{t\to\infty} \left(\int_{t_0}^{t} a(s) \left(\int_{t_0}^{s} b(r) \left(\int_{t_0}^{r} c(\tau)\Delta\tau\right)^{\beta} \Delta r\right)^{\alpha} \Delta s\right) \left(\int_{t}^{\infty} d(s)\Delta s\right)^{\gamma\beta\alpha} > 1. \tag{4.6}$$

Proof. Since (1.2) holds, every nonoscillatory solution of system (1.1) is of either Type (a) or Type (b) by Lemma 2.2. Assume (4.6) holds and (x, y, z, w) is of a Type (a) solution of system (1.1) such that $x(t) > 0$ for $t \geq T$. Let $\alpha\beta\gamma\lambda = 1$. Then, by (3.1) we have

$$I \left(\int_{t}^{\infty} d(s)\Delta s\right)^{\gamma\beta\alpha} \leq 1,$$

where I is given as in (3.2). Therefore,

$$\limsup_{t\to\infty} I \left(\int_{t}^{\infty} d(s)\Delta s\right)^{\gamma\beta\alpha} \leq 1$$

which contradicts (4.6). Therefore, (x, y, z, w) is of a Type (b) solution of system (1.1). □

Example 4.1. We consider the quantum time scale $\mathbb{T} = q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}\}$, where $q > 1$, $q \in \mathbb{R}$ and let $t_0 = 1$, $s = q^m$, and $t = q^n$ for $m, n \in \mathbb{N}_0$ for the system

$$\begin{cases} x^{\Delta}(t) = t^3 y^3(t) \\ y^{\Delta}(t) = \frac{1}{q} t z^3(t) \\ z^{\Delta}(t) = \frac{1}{q} t^8 w^5(t) \\ w^{\Delta}(t) = -\frac{1+q}{q^3 t^4} x(tq). \end{cases} \tag{4.7}$$

Then we have $\displaystyle\int_1^T t^3\Delta t = \sum_{t\in[1,T)_{q^{\mathbb{N}_0}}} t^3 t(q-1) = (q-1) \sum_{t\in[1,T)_{q^{\mathbb{N}_0}}} t^4$, and so $\displaystyle\int_1^\infty a(t)\Delta t = (q-1)\sum_{n=0}^{\infty} (q^4)^n = \infty$. It can be shown similarly that $\displaystyle\int_1^\infty b(t)\Delta t = \int_1^\infty c(t)\Delta t = \infty$. Also, $\displaystyle\int_1^T \frac{1+q}{q^3t^4}\Delta t = \frac{(q^2-1)}{q^3} \sum_{t\in[1,T)_{q^{\mathbb{N}_0}}} \frac{1}{t^3}$ implies $\int_1^\infty d(t)\Delta t = \frac{(q^2-1)}{q^3} \sum_{n=0}^{\infty} \frac{1}{(q^3)^n} < \infty$. Besides,

$$\begin{aligned} \int_1^T \frac{1+q}{q^3t^4} \left(\int_1^t s^3\Delta s\right) \Delta t &= \sum_{t\in[1,T)_{q^{\mathbb{N}_0}}} \frac{1+q}{q^3t^4} \left(\sum_{s\in[1,t)_{q^{\mathbb{N}_0}}} s^4(q-1)\right)(q-1)t \\ &= \frac{(1-q)}{(1+q^2)q^3} \sum_{t\in[1,T)_{q^{\mathbb{N}_0}}} \frac{1}{t^3}\left(1-t^4\right) \end{aligned}$$

and so

$$\int_1^\infty d(t)\left(\int_1^t a(s)\Delta s\right)\Delta t = \frac{(1-q)}{(1+q^2)q^3}\sum_{n=0}^\infty \left(\frac{1}{(q^3)^n} - q^n\right) = \infty$$

by geometric series. This shows that every nonoscillatory solution of system (4.7) is of a Type (b) by Theorem 4.1 (i). One can see that $\left(t, \frac{1}{t}, -\frac{1}{t}, \frac{1}{t^2}\right)$ is a nonoscillatory solution and hence it is of a Type (b) solution of system (4.7).

5. Conclusion

In this study, we present oscillation criteria for system (1.1). Condition (1.2) guarantees that any nonoscillatory solution (x, y, z, w) of system (1.1) is either of Type (a) or of Type (b), see Lemma 2.2. We show that system (1.1) is oscillatory when (2.1) holds. Then, we assume condition (2.2) instead of condition (2.1) to find oscillation criteria for system (1.1). In addition to condition (1.2), if (2.2) holds, Theorems 3.1 and 4.1 eliminate all Type (b) and Type (a) solutions of system (1.1), respectively. To achieve our goal, we use the integral conditions of the coefficient functions a, b, c and d and the product $\alpha\beta\gamma\lambda$. Furthermore, this discussion gives us the following theorem.

Theorem 5.1. *If one of the conditions of Theorem 3.1 and one of the conditions of Theorem 4.1 are assumed, then system* (1.1) *is oscillatory.*

We also observe that system (1.1) is oscillatory in the sub-linear, half-linear and super-linear cases.

Corollary 5.1. *System* (1.1) *satisfies the following:*

(i) *Assume Theorems 3.1* (iii) *and 4.1* (iii) *hold, then sub-linear system* (1.1) *is oscillatory.*
(ii) *Assume Corollary 3.1 and Theorem 4.1* (iv) *hold, then half-linear system* (1.1) *is oscillatory.*
(iii) *Assume Theorems 3.1* (iv) *and 4.1* (v) *hold, then super-linear system* (1.1) *is oscillatory.*

Note that an integral condition for a Type (a) solution in the half-linear system is still to be found.

As a consequence of our proofs, it is worth to mention that by the monotonicity of the first component all the results we have gotten in this study are also valid for the advanced system

$$\begin{cases} x^\Delta(t) = a(t)y^\alpha(t) \\ y^\Delta(t) = b(t)z^\beta(t) \\ z^\Delta(t) = c(t)w^\gamma(t) \\ w^\Delta(t) = -d(t)x^\lambda(k(t)), \end{cases}$$

where $k(t) \geq t$, $k \in C_{rd}([t_0, \infty)_{\mathbb{T}}, [t_0, \infty)_{\mathbb{T}})$ and $t \in \mathbb{T}$. At this point, one can consider the delay system

$$\begin{cases} x^{\Delta}(t) = a(t)y^{\alpha}(t) \\ y^{\Delta}(t) = b(t)z^{\beta}(t) \\ z^{\Delta}(t) = c(t)w^{\gamma}(t) \\ w^{\Delta}(t) = -d(t)x^{\lambda}(\tau(t)), \end{cases}$$

where $\tau(t) \leq t$, $\lim_{t\to\infty} \tau(t) = \infty$, and $\tau \in C_{rd}([t_0, \infty)_{\mathbb{T}}, [t_0, \infty)_{\mathbb{T}})$. Therefore, our question is now whether or not the same results are valid for the above delay system when (1.2) holds.

Note that without assuming (1.2), there are six more types of nonoscillatory solutions of system (1.1). As a result of this study, we also would like to find the oscillation conditions to eliminate other types.

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