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Solution Properties on Discrete Time Scales

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Dedicated to Allan Peterson on the occasion of his 60th birthday.

In this paper, we explore the solution properties of $u^{\Delta^2}(t) + p(t)u^{\gamma}(\sigma(t)) = 0$ on a time scale $\mathbb{T}$ with only isolated points, where $p(t)$ is defined on $\mathbb{T}$ and γ is a quotient of odd positive integers. Oscillation, nonoscillation, and solution comparisons, all depending on the sign of p, are included.

Keywords: Time scale; Oscillation; Discrete time scales; Nonoscillatory

AMS Subject Classifications: 39A13; 93C70

INTRODUCTION AND PRELIMINARIES

In this paper, we explore the solution properties of

$$u^{\Delta^2}(t) + p(t)u^{\gamma}(\sigma(t) = 0 \tag{1}$$

on a time scale $\mathbb{T}$ which contains only isolated points and is unbounded above, with the eventual goal of showing that if $\int_a^{\infty}\sigma(t)p(t)\Delta t = \infty$ then Eq. (1) is oscillatory. The function $p(t)$ is defined on $\mathbb{T}$ and γ is a quotient of odd positive integers. Some of the proof techniques in this paper are similar to those in a relatively recent book by Agarwal [1] on difference equations. An excellent resource for calculus on time scales is the Bohner and Peterson book [2].

By a solution $u(t)$ of the given dynamic equation we shall mean a nontrivial solution which exists on $[a, \infty)$ for some $a \in \mathbb{T}$. We now define oscillation and nonoscillation in this setting.

Definition 1 *A solution $u(t)$ is called oscillatory if for any $t_1 \in [a, \infty)$, there exists a $t_2 \in [t_1, \infty)$ such that $u(t_2)u(\sigma(t_2)) \leq 0$.*
The given dynamic equation itself is called oscillatory if all its solutions are oscillatory. If the solution $u(t)$ is not oscillatory, then it is said to be nonoscillatory. Equivalently the following definition can be made.

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DEFINITION 2 *The solution $u(t)$ is nonoscillatory if it is eventually positive or negative, i.e. there exists a $t_1 \in [a, \infty)$ such that $u(t)u(\sigma(t)) > 0$ for all $t \in [t_1, \infty)$.*
The given dynamic equation is called nonoscillatory if all of its solutions are nonoscillatory.

Example 1 A given dynamic equation can have both oscillatory and nonoscillatory solutions. Take

$$u^{\Delta^2}(t) + \frac{8}{3}u^{\Delta}(t) + \frac{4}{3}u(t) = 0$$

where $t \in \mathbb{T} = \mathbb{Z}$. Solutions to this difference equation are easily found (see Ref. [3]). Two solutions are $u_1(t) = (-1)^t$ and $u_2(t) = (1/3)^t$. Clearly $u_1(t)$ is oscillatory and $u_2(t)$ is nonoscillatory.
The sign properties of the exponential function on time scales are explored in Ref. [4].

Example 2 Let $\mathbb{T}$ be a time scale such that $\mu(t) \geq 1$ for all $t \in \mathbb{T}$. The dynamic equation

$$u^{\Delta^2}(t) + \frac{8}{3}u^{\Delta}(t) + \frac{4}{3}u(t) = 0$$

is regressive (see Ref. [2]). Then for $t_0 \in \mathbb{T}$, $e_{1/3}(t,t_0)$ and $e_{-1}(t,t_0)$ are two solutions of the above dynamic equation. However

$$e_{\frac{1}{3}}(t, t_0)e_{\frac{1}{3}}(\sigma(t), t_0) = \left(1 + \frac{1}{3}\mu(t)\right)\left(e_{\frac{1}{3}}(t, t_0)\right)^2 > 0$$

and

$$e_{-1}(t, t_0)e_{-1}(\sigma(t), t_0) = (1 - \mu(t)(e_{-1}(t, t_0)^2 \leq 0.$$

Thus $e_{1/3}(t,t_0)$ and $e_{-1}(t,t_0)$ are nonoscillatory and oscillatory solutions of above dynamic equation, respectively.

A VARIETY OF PROPERTIES OF SOLUTIONS

The following are some basic properties of solutions of Eq. (1).

LEMMA 1 *If $u(t)$ is a nontrivial solution of Eq. (1) with*

$$u(a)u(\sigma(a)) \leq 0$$

for some $a \in \mathbb{T}$, then either

$$u(a) \neq 0$$

or

$$u(\sigma(a)) \neq 0$$

Proof *Let $t = \rho(a)$ for $a \in \mathbb{T}$ and suppose $u(a) = 0$. We desire to show that $u(\sigma(a) \neq 0$. By Eq. (1) we have*

$$u^{\Delta^2}(\rho(a)) = 0,$$

or expanding

$$\frac{u^{\Delta}(a) - u^{\Delta}(\rho(a))}{\mu(\rho(a)} = 0$$

which implies that

$$\frac{u(\sigma(a)) - u(a)}{\mu(a)\mu(\rho(a))} - \frac{u(a) - u(\rho(a))}{\mu^2(\rho(a))} = 0.$$

However if both $u(a) = 0$ and $u(\sigma(a) = 0$, it must be the case that $u(\rho(a) = 0$. This process can be continued for $t = \rho^2(a)$, etc. implying that the solution $u(t)$ is actually trivial. But this contradicts the assumption that our solution is nontrivial. Similarly, if we assume $u(\sigma(a) = 0$, then it must be the case that $u(a) \neq 0$. Thus either $u(a) \neq 0$ or $u(\sigma(a) \neq 0$. □

Remark 1 If in addition, $a \in \mathrm{T}$, $u(a) = 0$, then

$$\mu(\rho(a)u(\sigma(a) = -\mu(a)u(\rho(a)).$$

Thus an oscillatory solution of Eq. (1) must change sign infinitely many times.

LEMMA 2 *Assume $p(t) \leq 0$ for all $t \in \mathbb{T}$, and for every $a \in \mathbb{T}$, $p(t) < 0$ for some $t \in [\sigma(a), \infty)$. If $u(t)$ is a solution of Eq. (1) with*

$$u(\rho(a)) \leq u(a) \tag{2}$$

and

$$u(a) \geq 0 \tag{3}$$

for some $a \in \mathbb{T}$, then $u(t)$ and $u^{\Delta}(t)$ are nondecreasing and nonnegative for all $t \in [a, \infty)$.

Proof *We will show the desired result by mathematical induction on t. Let $t = \rho(a)$ for $a \in \mathbb{T}$ in Eq. (1). Then by our assumption on p, Eqs. (2) and (3)*

$$u^{\Delta^2}(\rho(a)) = -p(\rho(a))u^{\gamma}(a) \geq 0 \tag{4}$$

and

$$u^{\Delta}(\rho(a)) = \frac{u(a) - u(\rho(a)}{\mu(\rho(a)} \geq 0.$$

It follows from Eq. (4) that

$$u^{\Delta^2}(\rho(a)) = \frac{u^{\Delta}(a) - u^{\Delta}(\rho(a))}{\mu(\rho(a))} \geq 0.$$

Therefore $u^{\Delta}(a) \geq u^{\Delta}(\rho(a)) \geq 0$. Suppose the desired result is true for $t = \sigma^{n-1}(a)$ for some $n \in \mathbb{N}$, $n > 1$, i.e.

$$u^{\Delta}(\sigma^n(a)) \geq u^{\Delta}(\sigma^{n-1}(a)) \geq 0 \tag{5}$$

and

$$u(\sigma^n(a)) \geq u(\sigma^{n-1}(a)) \geq 0. \tag{6}$$

We wish to show that the desired result is true for $t = \sigma(\sigma^{n-1}(a)) = \sigma^n(a)$ *for some* $n \in \mathbb{N}$, $n > 1$. *By Eq. (5),*

$$0 \leq u^{\Delta}(\sigma^n(a) = \frac{u(\sigma^{n+1}(a)) - u(\sigma^n(a))}{\mu(\sigma^n(a))}.$$

Because of this and by Eq. (6),

$$u(\sigma^{n+1}(a)) \geq u(\sigma^n(a)) \geq 0.$$

Therefore

$$u^{\Delta^2}(\sigma^n(a)) = -p(\sigma^n(a))u^{\gamma}(\sigma^{n+1}(a)) \geq 0.$$

Using

$$u^{\Delta^2}(\sigma^n(a)) = \frac{u^{\Delta}(\sigma^{n+1}(a)) - u^{\Delta}(\sigma^n(a))}{\mu(\sigma^n(a))} \geq 0$$

and Eq. (5),

$$u^{\Delta}(\sigma^{n+1}(a)) \geq u^{\Delta}(\sigma^n(a)) \geq 0.$$

Hence by induction the result holds. □

Remark 2 Similarly, if $p(t)$ is as in Lemma 2, $u(\rho(a)) \geq u(a)$, and $u(a) \leq 0$ for some $a \in \mathbb{T}$, then $u(t)$ and $u^{\Delta}(t)$ are nonincreasing and nonpositive for all $t \in [a, \infty)$.

The next lemma follows immediately from Lemma 2.

Lemma 3 *If* $p(t)$ *is as in Lemma 2, then all nontrivial solutions of Eq. (1) are nonoscillatory and eventually monotonic.*

Lemma 4 *Assume that* $p(t) \geq 0$ *for all* $t \in \mathbb{T}$, *and for every* $a \in \mathbb{T}$, $p(t) > 0$ *for some* $t \in [\sigma(a), \infty)$. *If* $u(t)$ *is a nonoscillatory solution of Eq. (1) such that*

$$u(t) > 0$$

for all $t \in [a, \infty)$, *then*

$$u(\sigma(t)) > u(t) \tag{7}$$

and

$$0 < u^{\Delta}(\sigma(t)) \leq u^{\Delta}(t) \tag{8}$$

for all $t \in [a, \infty)$.

Proof *If* $u(t)$ *is a nonoscillatory solution of Eq. (1), then since* $u(t) > 0$ *for* $t \in [a, \infty)$ *we have* $u(t)u(\sigma(t)) > 0$, *which implies that* $u(\sigma(t)) > 0$ *on* $[a, \infty)$ *as well. Thus*

$$u^{\Delta^2}(t) = -p(t)u^{\gamma}(\sigma(t)) \leq 0$$

on $[a, \infty)$. *Using*

$$u^{\Delta^2}(t) = \frac{u^{\Delta}(\sigma(t)) - u^{\Delta}(t)}{\mu(t)},$$

we have

$$\frac{u^{\Delta}(\sigma(t)) - u^{\Delta}(t)}{\mu(t)} \leq 0,$$

and so for $t \in [a, \infty)$

$$u^{\Delta}(\sigma(t)) \leq u^{\Delta}(t). \tag{9}$$

It remains to show that Eq. (7) holds which will imply that $0 < u^{\Delta}(\sigma(t))$. *Assume not, then we have* $u(\sigma(b)) \leq u(b)$ *for some* $b \in [\sigma(a), \infty)$. *By Eq. (9) we have*

$$0 \geq u^{\Delta}(b) \geq u^{\Delta}(\sigma(b)) \geq \cdots \geq u^{\Delta}(\sigma^n(b)) \geq \cdots. \tag{10}$$

However there exists a sequence $\{t_n\}_{n\in\mathbb{N}} \in \mathbb{T}$ *such that* $t_n \to \infty$ *and* $p(t_n) < 0$. *Thus*

$$u^{\Delta^2}(t_n) = -p(t_n)u^{\gamma}(\sigma(t_n)) < 0.$$

But

$$u^{\Delta^2}(t_n) = \frac{u^{\Delta}(\sigma(t_n)) - u^{\Delta}(t_n)}{\mu(t_n)} < 0,$$

so infinitely many of the inequalities in Eq. (10) must be strict, contradicting the fact that $u(t) > 0$ *for all* $t \in [a, \infty)$. □

Remark 3 If instead $u(t)$ is a nonoscillatory solution of Eq. (1) such that $u(t) < 0$ for all $t \in [a, \infty)$, then

$$u(\sigma(t)) < u(t)$$

and

$$0 > u^{\Delta}(\sigma(t)) \geq u^{\Delta}(t)$$

for all $t \in [a, \infty)$.

Remark 4 For $a, t \in \mathbb{T}$, $t > a$ we can write $t = \sigma^n(a)$ for some $n \in \mathbb{N}$. Thus we can write

$$t - \sigma(a) = \sigma^n(a) - \sigma(a) = \sum_{i=1}^{n-1} \mu(\sigma^i(a)).$$

If instead $t < a$ we can write $t = \rho^n(a)$ for some $n \in \mathbb{N}$, so

$$\sigma(a) - t = \sigma(a) - \rho^n(a) = \sum_{i=0}^{n} \mu(\rho^i(a)).$$

THEOREM 1 *Assume* $p(t) \leq 0$ *for all* $t \in \mathbb{T}$, *and for every* $a \in \mathbb{T}$, $p(t) < 0$ *for some* $t \in [\sigma(a), \infty)$ *and for some* $t \in (-\infty, \rho(a)]$. *Let* $u(t)$ *and* $v(t)$ *be solutions of Eq. (1) satisfying*

$$u(b) \leq v(b) \tag{11}$$

and

$$u(\sigma(b)) > v(\sigma(b)) \tag{12}$$

for some $b \in \mathbb{T}$. *Then for* $t \in [\sigma(b), \infty)$,

$$u(t) - v(t) \geq \frac{t-b}{\mu(b)}[u(\sigma(b)) - v(\sigma(b))], \tag{13}$$

and for $t \in (-\infty, b]$

$$u(t) - v(t) \leq \frac{\sigma(b) - t}{\mu(b)}[u(b) - v(b)]. \tag{14}$$

In addition

$$u(t) > v(t)$$

for all $t \in [\sigma(b), \infty)$,

$$u(t) < v(t)$$

for all $t \in (-\infty, \rho(b)]$, *and* $u(t) - v(t)$ *is nondecreasing for all* $t \in \mathbb{T}$.

Proof Fix $r \in \mathbb{T}$, $r > b$, and let $w(\sigma^n(r)) = u(\sigma^n(b)) - v(\sigma^n(b))$ for $n \in \mathbb{N}_0$. From Eqs. (11) and (12) it is clear that $w(r) = u(b) - v(b) \leq 0$ and $w(\sigma(r) = u(\sigma(b)) - v(\sigma(b)) > 0$. By induction we shall show that

$$w(\sigma^n(r)) \geq \frac{\sum_{i=0}^{n-1} \mu(\sigma^i(b))}{\sum_{i=0}^{n-2} \mu(\sigma^i(b))} w(\sigma^{n-1}(r)) > 0 \tag{15}$$

where $n \in \mathbb{N}$, $n \geq 2$. From Eq. (1) we have

$$u^{\Delta^2}(b) = -p(b)u^{\gamma}(\sigma(b)) \geq -p(b)v^{\gamma}(\sigma(b)) = v^{\Delta^2}(b)$$

it follows that for $n = 2$, $t = \sigma^2(b)$,

$$\begin{aligned}
w(\sigma^2(r)) &= u(\sigma^2(b)) - v(\sigma^2(b)) \\
&= u(\sigma(b)) + \mu(\sigma(b))u^{\Delta}(\sigma(b)) - v(\sigma(b)) - \mu(\sigma(b))v^{\Delta}(\sigma(b)) \\
&= w(\sigma(r)) + \mu(\sigma(b))[u^{\Delta}(\sigma(b)) - v^{\Delta}(\sigma(b))] \\
&= w(\sigma(r)) + \mu(\sigma(b))[u^{\Delta}(b) + \mu(b)u^{\Delta^2}(b) - v^{\Delta}(b) - \mu(b)v^{\Delta^2}(b)] \\
&\geq w(\sigma(r)) + \mu(\sigma(b))\left[\frac{w(\sigma(r)) - w(r)}{\mu(b)}\right] \geq w(\sigma(r)) + w(\sigma(r))\frac{\mu(\sigma(b))}{\mu(b)} \\
&= \frac{\mu(b) + \mu(\sigma(b))}{\mu(b)} w(\sigma(r)) > 0.
\end{aligned}$$

Hence Eq. (15) is true for $n = 2$. Now suppose that Eq. (15) is true for some $n \geq 2$. We wish to show Eq. (15) holds for $n + 1$. As before we have that

$$u^{\Delta^2}(\sigma^{n-1}(b)) \geq v^{\Delta^2}(\sigma^{n-1}(b)).$$

Hence

$$\begin{aligned}
w(\sigma^{n+1}(r)) &= u(\sigma^{n+1}(b)) - v(\sigma^{n+1}(b)) \\
&= u(\sigma^n(b)) + \mu(\sigma^n(b))u^{\Delta}(\sigma^n(b)) - v(\sigma^n(b)) - \mu(\sigma^n(b))v^{\Delta}(\sigma^n(b)) \\
&= w(\sigma^n(r)) + \mu(\sigma^n(b))[u^{\Delta}(\sigma^n(b)) - v^{\Delta}(\sigma^n(b))] \\
&= w(\sigma^n(r)) + \mu(\sigma^n(b))[u^{\Delta}(\sigma^{n-1}(b)) + \mu(\sigma^{n-1}(b))u^{\Delta^2}(\sigma^{n-1}(b)) \\
&\quad - v^{\Delta}(\sigma^{n-1}(b)) - \mu(\sigma^{n-1}(b))v^{\Delta^2}(\sigma^{n-1}(b))] \\
&\geq w(\sigma^n(r)) + \mu(\sigma^n(b))\left[\frac{w(\sigma^n(r)) - w(\sigma^{n-1}(r))}{\mu(\sigma^{n-1}(b))}\right] \\
&\geq w(\sigma^n(r)) + w(\sigma^n(r))\frac{\mu(\sigma^n(b))}{\mu(\sigma^{n-1}(b))} - \frac{\mu(\sigma^n(b))}{\mu(\sigma^{n-1}(b))}\frac{\sum_{i=0}^{n-2}\mu(\sigma^i(b))}{\sum_{i=0}^{n-1}\mu(\sigma^i(b))}w(\sigma^n(r)) \\
&= \left[1 + \frac{\mu(\sigma^n(b))}{\mu(\sigma^{n-1}(b))} - \frac{\mu(\sigma^n(b))}{\mu(\sigma^{n-1}(b))}\frac{\sum_{i=0}^{n-2}\mu(\sigma^i(b))}{\sum_{i=0}^{n-1}\mu(\sigma^i(b))}\right]w(\sigma^n(r)) \\
&= \frac{\sum_{i=0}^{n}\mu(\sigma^i(b))}{\sum_{i=0}^{n-1}\mu(\sigma^i(b))}w(\sigma^n(r)) > 0.
\end{aligned}$$

Thus Eq. (15) holds for $n+1$ *as well. From Eqs. (12) and (15), it is clear that* $u(t) > v(t)$ *for all* $t \in [\sigma(b), \infty)$. *Further we have*

$$\begin{aligned}
w(\sigma^n(r)) &\geq \frac{\sum_{i=0}^{n-1}\mu(\sigma^i(b))}{\sum_{i=0}^{n-2}\mu(\sigma^i(b))}w(\sigma^{n-1}(r)) \\
&\geq \frac{\sum_{i=0}^{n-1}\mu(\sigma^i(b))\sum_{i=0}^{n-2}\mu(\sigma^i(b))}{\sum_{i=0}^{n-2}\mu(\sigma^i(b))\sum_{i=0}^{n-3}\mu(\sigma^i(b))}w(\sigma^{n-2}(r)) \\
&= \frac{\sum_{i=0}^{n-1}\mu(\sigma^i(b))}{\sum_{i=0}^{n-3}\mu(\sigma^i(b))}w(\sigma^{n-2}(r)) \geq \ldots \geq \frac{\sum_{i=0}^{n-1}\mu(\sigma^i(b))}{\mu(b)}w(\sigma(r)) \\
&= \frac{\sigma^n(b) - b}{\mu(b)}w(\sigma(r))
\end{aligned}$$

which is the same as Eq. (13) for $t = \sigma^n(b)$.
For the last part of the theorem, we let $w(\rho^n(r)) = u(\rho^n(b)) - v(\rho^n(b))$ *for* $n \in \mathbb{N}_0$. *By Eq. (1) we have*

$$u^{\Delta^2}(\rho(b)) = -p(\rho(b))u^{\gamma}(b) \leq -p(\rho(b))v^{\gamma}(b) = v^{\Delta^2}(\rho(b)).$$

In addition $w(r) = u(b) - v(b) \le 0$ *and* $w(\sigma(r)) = u(\sigma(b)) - v(\sigma(b)) > 0$. *For* $t = \rho(r)$ *we have*

$$\begin{aligned}
w(\rho(r)) &= u(\rho(b)) - v(\rho(b)) = u(b) - \mu(\rho(b))u^{\Delta}(\rho(b)) \\
&\quad - v(b) + \mu(\rho(b))v^{\Delta}(\rho(b)) \\
&= w(r) - \mu(\rho(b))[u^{\Delta}(b) - \mu(\rho(b))u^{\Delta^2}(\rho(b))] \\
&\quad + \mu(\rho(b))[v^{\Delta}(b) - \mu(\rho(b))v^{\Delta^2}(\rho(b))] \\
&= w(r) - \mu(\rho(b))[u^{\Delta}(b) - v^{\Delta}(b)] \\
&\quad + \mu(\rho(b))\mu(\rho(b))[u^{\Delta^2}(\rho(b)) - v^{\Delta^2}(\rho(b))] \\
&\le w(r) - \mu(\rho(b))[u^{\Delta}(b) - v^{\Delta}(b)] \\
&= w(r) - \frac{\mu(\rho(b))}{\mu(b)}w(\sigma(r)) + \frac{\mu(\rho(b))}{\mu(b)}w(r) \\
&= \frac{\mu(\rho(b)) + \mu(b)}{\mu(b)}w(r) - \frac{\mu(\rho(b))}{\mu(b)}w(\sigma(r)) \\
&< \frac{\mu(\rho(b)) + \mu(b)}{\mu(b)}w(r) \le 0,
\end{aligned}$$

so $w(\rho(r)) < 0$ *as well. We shall show that*

$$w(\rho^n(r)) < \frac{\sum_{i=0}^{n} \mu(\rho^i(b))}{\sum_{i=0}^{n-1} \mu(\rho^i(b))} w(\rho^{n-1}(r)) < 0 \tag{16}$$

where $n \ge 2$. *Using the same relationships as previous in the proof we have*

$$\begin{aligned}
w(\rho^2(r)) &= u(\rho^2(b)) - v(\rho^2(b)) \le w(\rho(r)) - \mu(\rho^2(b))[u^{\Delta}(\rho(b)) - v^{\Delta}(\rho(b))] \\
&= w(\rho(r)) - \frac{\mu(\rho^2(b))}{\mu(\rho(b))}w(r) + \frac{\mu(\rho^2(b))}{\mu(\rho(b))}w(\rho(r)) \\
&= \frac{\mu(\rho^2(b)) + \mu(\rho(b))}{\mu(\rho(b))}w(\rho(r)) - \frac{\mu(\rho^2(b))}{\mu(\rho(b))}w(r) \\
&< \frac{\mu(\rho^2(b)) + \mu(\rho(b)) + \mu(b)}{\mu(\rho(b)) + \mu(b)}w(\rho(r)) < 0
\end{aligned}$$

so Eq. (16) is true for $n = 2$. *Suppose Eq. (16) is true for* $n \ge 2$, *then we wish to show it is true for* $n + 1$. *As before*

$$u^{\Delta^2}(\rho^{n+1}(b)) \le v^{\Delta^2}(\rho^{n+1}(b)).$$

Thus

$$
\begin{aligned}
w(\rho^{n+1}(r)) &= u(\rho^{n+1}(b)) - v(\rho^{n+1}(b)) \\
&= u(\rho^n(b)) - \mu(\rho^{n+1}(b))u^{\Delta}(\rho^{n+1}(b)) - v(\rho^n(b)) + \mu(\rho^{n+1}(b))v^{\Delta}(\rho^{n+1}(b)) \\
&= w(\rho^n(r)) - \mu(\rho^{n+1}(b))[u^{\Delta}(\rho^n(b)) - v^{\Delta}(\rho^n(b))] + \mu(\rho^{n+1}(b))\mu(\rho^{n+1}(b)) \\
&\quad \times [u^{\Delta^2}(\rho^{n+1}(b)) - v^{\Delta^2}(\rho^{n+1}(b))] \\
&\leq w(\rho^n(r)) - \mu(\rho^{n+1}(b))[u^{\Delta}(\rho^n(b)) - v^{\Delta}(\rho^n(b))] \\
&= w(\rho^n(r)) - \frac{\mu(\rho^{n+1}(b))}{\mu(\rho^n(b))} w(\rho^{n-1}(r) + \frac{\mu(\rho^{n+1}(b))}{\mu(\rho^n(b)))} w(\rho^n(r)) \\
&< \frac{\mu(\rho^n(b)) + \mu(\rho^{n+1}(b))}{\mu(\rho^n(b))} w(\rho^n(r)) - \frac{\mu(\rho^{n+1}(b))\sum_{i=0}^{n-1}\mu(\rho^i(b))}{\mu(\rho^n(b))\sum_{i=0}^{n}\mu(\rho^i(b))} w(\rho^n(r)) \\
&= \frac{\sum_{i=0}^{n+1}\mu(\rho^i(b))}{\sum_{i=0}^{n}\mu(\rho^i(b))} w(\rho^n(r)) < 0,
\end{aligned}
$$

and Eq. (16) holds for $n+1$. As before we can use Eq. (16) to obtain

$$w(\rho^n(r)) < \frac{\sigma(b)) - \rho^n(b)}{\mu(b)} w(r)$$

for $n \in \mathbb{N}$, which is equivalent to Eq. (14). In addition $u(t) - v(t)$ is nondecreasing on $\mathbb{T}$ and $u(t) < v(t)$ for all $t \in (-\infty, \rho(b)]$. □

Remark 5 In the case $\mathbb{T} = \mathbb{Z}$, Eq. (13) reduces to

$$u(t) - v(t) \geq (t - b)(u(b+1) - v(b+1),$$

for $t \in [b+1, \infty)$. In addition for $t \in (-\infty, b]$, Eq. (14) reduces to

$$u(t) - v(t) \leq (b + 1 - t)(u(b) - v(b))$$

which is as expected from Ref. [1].

Remark 6 In Lemma 2 we assumed that $u(a) \geq u(\rho(a))$, $u(a) \geq 0$, and concluded that $u(t)$ was nondecreasing for all $t \in [a, \infty)$. If we assume that $u(a) > u(\rho(a)) \geq 0$, then $u(t)$ is strictly increasing on $[\rho(a), \infty)$ and $u(t) \to \infty$ as $t \to \infty$.

Proof *By assumption $u^{\Delta}(\rho(a)) > 0$. Using Lemma 2 $u^{\Delta}(t)$ is nondecreasing, but this implies that $u^{\Delta}(t) > 0$ for $t \in [\rho(a), \infty)$. Thus $u(t)$ is strictly increasing on $[\rho(a), \infty)$. Let $z(t)$ be a solution of Eq. (1) defined by*

$$z(a) = z(\rho(a)) = u(\rho(a)).$$

By Lemma 2, $z(t)$ is nonnegative on $[a, \infty)$. Now apply Theorem (1) with $b = \rho(a)$. Since $u(b) = z(b)$ and $u(\sigma(b)) > z(\sigma(b))$, we have from Theorem (1) that

$$u(t) \geq u(t) - z(t) \geq \frac{t - \rho(a)}{\mu(\rho(a))}(u(a) - z(a) = \frac{t - \rho(a)}{\mu(\rho(a))}(u(a) - u(\rho(a)))$$

where $u(a) - u(\rho(a)) > 0$. Thus $u(t) \to \infty$ as $t \to \infty$. □

The following Corollary is a direct result of Theorem 1.

COROLLARY 1 *If $p(t)$ is as in Lemma 2 and $u(t)$, $v(t)$ are solutions of Eq. (1) satisfying $u(a) = v(a)$ and $u(b) = v(b)$ for some $a < b$, $a, b \in \mathbb{T}$, then $u(t) = v(t)$ for all $t \in \mathbb{T}$.*

LEMMA 5 *If $p(t)$ is as in Lemma 2, then for any $\sigma(a) > b$, $a, b \in \mathbb{T}$, there exists a unique solution of Eq. (1) such that $u(b) = u_0$ and $u(\sigma(a)) = 0$, where u_0 is any positive constant.*

Proof Let $z(t)$ be a solution of Eq. (1) such that $z(\sigma(a)) = 0$. If $z(a) > 0$ and $z(\rho(a)) \leq z(a)$, then Lemma 2 implies that $z(\sigma(a)) \geq z(a) > 0$, which is a contradiction. Thus $z(\rho(a)) > z(a) > 0$. Proceeding in this way we obtain

$$z(b) > z(\sigma(b)) > \cdots > z(a) > z(\sigma(a)) = 0. \tag{17}$$

Since $z(\sigma(a)) = 0$, if $z(a)$ is also specified then $z(t)$ is uniquely determined for all $t \in [b, \sigma(a)]$ by Eq. (1). Thus in particular $z(b)$) is determined by $z(a)$. Let f be the mapping from $z(a)$ to $z(b)$. From Eq. (1) it is clear that each $z(t)$, $t \in [b, \rho(a)]$ continuously depends on $z(a)$, and so the function $z(b) = f(z(a))$ is continuous. If we let $z(a) = u_0$, then Eq. (17) implies that $f(u_0) > u_0$; if we let $z(a) = 0$, so that $z(\sigma(a)) = z(a) = 0$, then $z(t) = 0$ by Lemma 2, so $f(0) = 0$. Thus since f is continuous, there exists a β, $0 < \beta < u_0$, such that $f(\beta) = u_0$. Therefore there exists a solution $u(t)$ of Eq. (1) determined by $u(\sigma(a)) = 0$ and $u(a) = \beta$ which must satisfy $u(b) = u_0$. Finally the uniqueness of the solution follows from Corollary 1. □

THEOREM 2 *If $p(t)$ is as in Lemma 2, then Eq. (1) has a positive nonincreasing solution $u(t)$ and a positive strictly increasing solution $v(t)$ such that $v(t) \to \infty$ as $t \to \infty$. In addition, the nonincreasing solution $u(t)$ is uniquely determined once $u(a)$ is specified.*

Proof If we choose $a \in \mathbb{T}$, $v(a) = 1$ and $v(\sigma(a)) > 1$, then the existence of an increasing solution $v(t)$ satisfying the stated properties is an immediate consequence of Remark 6. We wish to show the existence of a positive nonincreasing solution $u(t)$. It is clear from Lemma 5 that for each $n \in \mathbb{T}$, $n \geq \max\{1, \sigma(a)\}$, there is a unique solution $u_n(t)$, $t \in \mathbb{T}$ of Eq. (1) such that

$$u_n(a) = u_a, \quad u_n(n) = 0. \tag{18}$$

Further, in view of Eq. (17) we know that for every $n \geq \max\{1, \sigma(a)\}$,

$$u_a \geq u_n(t) > u_n(\sigma(t)) \geq 0 \quad \text{for} \quad t \in [a, \rho(n)]. \tag{19}$$

We claim that for every $n \geq \max\{1, \sigma(a)\}$,

$$u_{\sigma(n)}(t) > u_n(t) \quad \text{for} \quad t \in [\sigma(a), \infty). \tag{20}$$

For this, by Theorem 1 it suffices to show that

$$u_{\sigma(n)}(\sigma(a)) > u_n(\sigma(a)).$$

By way of contradiction suppose that $u_{\sigma(n)}(\sigma(a)) \leq u_n(\sigma(a))$. If $u_{\sigma(n)}(\sigma(a)) = u_n(\sigma(a))$, then since $u_n(a) = u_{\sigma(n)}(a) = u_a$, the solutions $u_n(t)$ and $u_{\sigma(n)}(t)$ are identically equal. However $u_{\sigma(n)}(\sigma(n)) = u_n(n) = 0$, so both $u_n(t)$ and $u_{\sigma(n)}(t)$ are identically zero, which contradicts

$u_n(a) = u_a > 0$. On the other hand, if $u_{\sigma(n)}(\sigma(a)) < u_n(\sigma(a))$ then from Theorem 1 we have $u_n(t) > u_{\sigma(n)}(t)$ for all $t \in [\sigma(a), \infty)$. In particular for $t = n$ we find

$$0 = u_n(n) > u_{\sigma(n)}(n) > u_{\sigma(n)}(\sigma(n)) = 0,$$

which is also a contradiction. Hence Eq. (20) holds.

Combining Eqs. (19) and (20) we find for each $t \in [\sigma(a), \infty)$, the sequence $\{u_n(t)\}_{n\in\mathbb{T}}$ is increasing, bounded above by u_a, and is eventually positive. For each $t \in \mathbb{T}$, let

$$u(t) = \lim_{n\to\infty} u_n(t).$$

Then $0 < u(t) \le u_a$ for $t \in \mathbb{T}$, and from Eq. (19) we have $u(t) \ge u(\sigma(t))$. Now since $n \in [\sigma(a), \infty)$, $u_n(t)$ is a solution of Eq. (1), and we have

$$(u_n(t))^{\Delta^2} = -p(t)(u_n(\sigma(t)))^{\gamma}.$$

Thus as $n \to \infty$, we find that $u(t)$ is a nonincreasing positive solution of Eq. (1).

Finally we show that the solution $u(t)$ is unique once u_a is specified. For this let $z(t)$ be another positive nonincreasing solution of Eq. (1) such that $z(a) = u_a$. Then either $z(\sigma(a)) < u(\sigma(a))$, $z(\sigma(a)) > u(\sigma(a))$, or $z(\sigma(a)) = u(\sigma(a))$.

If $z(\sigma(a)) < u(\sigma(a))$, then there exists a $n \in \mathbb{T}$ and a solution $u_n(t)$ defined by Eq. (18) such that

$$z(\sigma(a)) < u_n(\sigma(a)) < u(\sigma(a)).$$

Since $u_n(a) = z(a)$ and $u_n(\sigma(a)) > z(\sigma(a))$, Theorem 1 implies that $u_n(t) > z(t)$ for all $t \in [\sigma(a), \infty)$. In particular this implies $0 = u_n(n) > z(n)$, which is a contradiction. If instead $z(\sigma(a)) > u(\sigma(a))$ then Theorem (1) implies that

$$z(t) - u(t) \ge \frac{t-a}{\mu(a)}(z(\sigma(a)) - u(\sigma(a))), \quad t \in [\sigma(a), \infty).$$

This means that $z(t)$ becomes unbounded as $t \to \infty$ since $(t-a)/\mu(a) \to \infty$ as $t \to \infty$, which is again a contradiction. Thus $z(\sigma(a)) = u(\sigma(a))$. By Corollary 1 $z(t) = u(t)$ for all $t \in \mathbb{T}$. □

THEOREM 3 *Let $p(t)$ be as in Lemma 4, $a \in \mathbb{T}$ $a \ge 0$, and $\gamma > 1$. If*

$$\int_a^{\infty} \sigma(t)p(t)\Delta t = \infty,$$

then the dynamic Eq. (1) is oscillatory.

Proof *Let $u(t)$ be a nonoscillatory solution of Eq. (1), and $u(t) > 0$ for all $t \in [a, \infty)$. Multiply both sides of Eq. (1) by $\sigma(t)u^{-\gamma}(\sigma(t))$ to obtain*

$$\sigma(t)u^{-\gamma}(\sigma(t))u^{\Delta^2}(t) + \sigma(t)p(t) = 0.$$

Using the integration by parts formula for $k \in [a, \infty)$

$$\int_a^k \sigma(t)u^{-\gamma}(\sigma(t))u^{\Delta^2}(t)\Delta t = ku^{-\gamma}(k)u^{\Delta}(k) - au^{-\gamma}(a)u^{\Delta}(a) - \int_a^k (tu^{-\gamma}(t))^{\Delta}u^{\Delta}(t)\Delta t$$

yields

$$ku^{-\gamma}(k)u^{\Delta}(k) - au^{-\gamma}(a)u^{\Delta}(a) - \int_a^k (tu^{-\gamma}(t))^{\Delta}u^{\Delta}(t)\Delta t + \int_a^k \sigma(t)p(t)\Delta t = 0.$$

In view of Lemma 4 and the hypothesis, it must be the case that

$$\int_a^k (tu^{-\gamma}(t))^{\Delta}u^{\Delta}(t)\Delta t \to \infty \text{ as } k \to \infty. \tag{21}$$

We shall show that Eq. (21) is impossible.
Note that $u^{\Delta}(t) > 0$ *implies* $(u^{-\gamma}(t))^{\Delta} < 0$. *Thus*

$$\int_a^k (tu^{-\gamma}(t))^{\Delta}u^{\Delta}(t)\Delta t = \int_a^k [u^{-\gamma}(\sigma(t)) + t(u^{-\gamma}(t))^{\Delta}]u^{\Delta}(t)\Delta t \le \int_a^k u^{-\gamma}(\sigma(t))u^{\Delta}(t)\Delta t$$

and it suffices to show that

$$\int_a^k u^{-\gamma}(\sigma(t))u^{\Delta}(t)\Delta t < \infty. \tag{22}$$

We define $r(s)$*, a continuous function on* $[t, \sigma(t)]$ *by*

$$r(s) = u(t) + (s - t)u^{\Delta}(t).$$

Notice that $r(t) = u(t)$, $r(\sigma(t)) = u(\sigma(t))$, *and* $r'(s) = u^{\Delta}(t) > 0$. *Hence* $r(s)$ *is continuous and increasing for* $s \in [t, \sigma(t)]$. *From this we get*

$$\begin{aligned} u^{-\gamma}(\sigma(t))u^{\Delta}(t) &= \frac{1}{\mu(t)}\int_t^{\sigma(t)} u^{-\gamma}(\sigma(t))u^{\Delta}(t)\mathrm{d}s = \frac{1}{\mu(t)}\int_t^{\sigma(t)} r^{-\gamma}(\sigma(t))r'(s)\mathrm{d}s \\ &\le \frac{1}{\mu(t)}\int_t^{\sigma(t)} r^{-\gamma}(s)r'(s)\mathrm{d}s = \frac{1}{\mu(t)}\frac{1}{1-\gamma}[r^{1-\gamma}(\sigma(t)) - r^{1-\gamma}(t)] \\ &= \frac{1}{1-\gamma}\frac{[r^{1-\gamma}(\sigma(t)) - r^{1-\gamma}(t)]}{\mu(t)} = \frac{1}{1-\gamma}(r^{1-\gamma})^{\Delta}(t). \end{aligned}$$

This implies that for $k \in \mathbb{T}$,

$$\int_a^k u^{-\gamma}(\sigma(t))u^{\Delta}(t)\Delta t \le \frac{1}{1-\gamma}(r^{1-\gamma}(k) - r^{1-\gamma}(a)).$$

However since $\gamma > 1$ *and* r *is an increasing function, it follows that Eq. (22) holds, completing the proof.* □

There are still many more possible results for Eq. (1) regarding oscillation; such as whether or not $u(t)$ being an oscillatory solution implies that $\int_a^{\infty}\sigma(t)p(t)\Delta t = \infty$, and how $0 < \gamma < 1$ affects the results. However that is a subject for later work.

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