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Oscillation Criteria for Certain Fourth-Order Nonlinear Delay Differential Equations

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Abstract. In this article, we establish some new criteria for the oscillation of fourth-order nonlinear delay differential equations of the form

$$(r_2(t)(r_1(t)(y''(t))^\alpha)')' + p(t)(y''(t))^\alpha + q(t)f(y(g(t))) = 0$$

provided that the second-order equation

$$(r_2(t)z'(t))') + \frac{p(t)}{r_1(t)}z(t) = 0$$

is nonoscillatory or oscillatory.

Mathematics Subject Classification. 34C10, 39A10.

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1. Introduction

In this article, we consider nonlinear fourth-order functional differential equations of the form

$$(r_2(t)(r_1(t)(y''(t))^\alpha)')' + p(t)(y''(t))^\alpha + q(t)f(y(g(t))) = 0, \quad t \geq t_0 > 0, \tag{1.1}$$

where $\alpha \geq 1$ is the ratio of positive odd integers. We assume that

(i) $r_1, r_2 \in C([t_0, \infty), \mathbb{R}^+), \mathbb{R}^+ = (0, \infty)$,
(ii) $p, q \in C([t_0, \infty), \mathbb{R}^+)$,
(iii) $g \in C^1([t_0, \infty), \mathbb{R}), g'(t) \geq 0, g(t) \to \infty$ as $t \to \infty$,
(iv) $f \in C(\mathbb{R}, \mathbb{R}), xf(x) > 0$ and $f(x)/x^\beta \geq k > 0, k$ is a constant, for $x \neq 0$, where β is the ratio of positive odd integers.

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We restrict our attention to those solutions of Eq. (1.1) which exist on $I = [t_0, \infty)$ and satisfy the condition

$$\sup\{|y(t)| : t_1 \le t < \infty\} > 0 \quad \text{for } t_1 \in [t_0, \infty).$$

Such a solution is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. Equation (1.1) is said to be oscillatory if it has an oscillatory solution.

In the last three decades, there has been an increasing interest in studying oscillation and nonoscillation of solutions of functional differential equations. Most of the work on this subject, however, has been restricted to first- and second-order equations as well as equations of type (1.1) when $\alpha = 1$, $p(t) = 0$ and other higher-order equations. For recent contributions, we refer to [1–16]. It appears that little is known regarding the oscillation of Eq. (1.1). Therefore, our main goal is to establish some new criteria for the oscillation of all solutions of Eq. (1.1).

Using a generalized Riccati transformation, integral averaging technique and comparison with first-order delay equations, we shall establish some sufficient conditions which insure that any solution of Eq. (1.1) oscillates when the associated equation

$$(r_2(t)z'(t))') + \frac{p(t)}{r_1(t)}z(t) = 0$$

is nonoscillatory or oscillatory.

2. Main Results

For the sake of brevity, we define

$$L_0y(t) = y(t), \quad L_1y(t) = y'(t), \quad L_2y(t) = r_1(t)((L_0y(t))'')^\alpha,$$

$$L_3y(t) = r_2(t)(L_2y(t))', \quad L_4y(t) = (L_3y(t))', \quad t \in [t_0, \infty).$$

Then, Eq. (1.1) can be written as

$$L_4y(t) + \frac{p(t)}{r_1(t)}L_2y(t) + q(t)f(y(g(t))) = 0.$$

Remark 2.1. If y is a solution of Eq. (1.1), then $z = -y$ is a solution of the equation

$$L_4z(t) + \frac{p(t)}{r_1(t)}L_2z(t) + q(t)f^*(z(g(t))) = 0,$$

where $f^*(z) = -f(-z)$ and $zf^*(z) > 0$ for $z \ne 0$. Thus, concerning nonoscillatory solution of Eq. (1.1), we can restrict our attention only to the positive ones.

Define the functions

$$R_1(t, t_1) = \int_{t_1}^{t} r_1^{-1/\alpha}(s)\mathrm{d}s, \quad R_2(t, t_1) = \int_{t_1}^{t} \frac{\mathrm{d}s}{r_2(s)},$$

$$R_{12}(t,t_1) = \int_{t_1}^{t} \left(\frac{1}{r_1(s)} R_2(s,t_1)\right)^{1/\alpha} \mathrm{d}s, \quad R_{12}^*(t,t_1)$$
$$= \int_{t_2}^{t} \int_{t_2}^{u} \left(\frac{1}{r_1(s)} R_2(s,t_1)\right)^{1/\alpha} \mathrm{d}s\mathrm{d}u,$$

for $t_0 \le t_1 \le t < \infty$.

We assume that

$$R_1(t,t_0) \to \infty \text{ as } t \to \infty, \tag{2.1}$$
$$R_2(t,t_0) \to \infty \text{ as } t \to \infty. \tag{2.2}$$

In this section, we state and prove the following lemmas which we will use in the proof of our main results.

Lemma 2.1. *Assume that*

$$(r_2(t)z'(t))' + \frac{p(t)}{r_1(t)} z(t) = 0 \tag{2.3}$$

is nonoscillatory. If y is a nonoscillatory solution of Eq. (1.1) *on* $[t_1,\infty), t_1 \ge t_0$*, then there exists a* $t_2 \in [t_1,\infty)$ *such that* $y(t)L_2(y(t)) > 0$ *or* $y(t)L_2(y(t)) < 0$ *for* $t \ge t_2$.

Proof. Let y be a nonoscillatory solution of Eq. (1.1) on $[t_1,\infty)$, say $y(t) > 0$ and $y(g(t)) > 0$ for $t \ge t_1 \ge t_0$. Set $x(t) = -L_2y(t)$. From Eq. (1.1), the function $x(t)$ satisfies the equation

$$(r_2(t)x'(t))' + \left(\frac{p(t)}{r_1(t)}\right) x(t) = q(t)f(y(g(t))) > 0, \quad t \ge t_1. \tag{2.4}$$

We claim that all solutions of Eq. (2.4) are nonoscillatory. Let u be a solution of Eq. (2.3), say $u(t) > 0$ for $t \ge t_1 \ge t_0$. Note that if $u(t)$ is negative, then $-u(t)$ is also a solution of Eq. (2.3).

Let $x(t)$ be oscillatory and have consecutive zeros at a and b ($t_1 < a < b$) such that $x'(a) \ge 0, x'(b) \le 0$ and $x(t) \ge 0$ for $t \in (a,b)$. Multiplying Eq. (2.4) by $u(t)$ and integrating over $[a,b]$, we obtain

$$r_2(b)x'(b)u(b) - r_2(a)x'(a)u(a) - \int_a^b (r_2(t)u'(t))x'(t)\mathrm{d}t + \int_a^b \frac{p(t)}{r_1(t)} u(t)x(t)\mathrm{d}t > 0.$$

Integrating by parts again and using that $x(a) = 0$ and $x(b) = 0$, we get

$$r_2(b)x'(b)u(b) - r_2(a)x'(a)u(a) + \int_a^b \left((r_2(t)u'(t))' + \frac{p(t)}{r_1(t)} u(t)\right) x(t)\mathrm{d}t > 0.$$

Thus, we have a contradiction. This completes the proof. □

Lemma 2.2. *Let y be a solution of Eq.* (1.1) *with* $y(t)L_2y(t) > 0$ *for* $t \ge t_1 \ge t_0$*. Then,*

$$L_2y(t) > R_2(t,t_1)L_3y(t), \quad t \ge t_1, \tag{2.5}$$
$$L_1y(t) > R_{12}(t,t_1)L_3^{1/\alpha}y(t), \quad t \ge t_1, \tag{2.6}$$

and

$$y(t) > R_{12}^*(t,t_1)L_3^{1/\alpha}y(t), \quad t \ge t_1. \tag{2.7}$$

Proof. Let y be a solution of Eq. (1.1), say $y(t) > 0,\ y(g(t)) > 0$ and $L_2y(t) > 0$ for $t \geq t_1 \geq t_0$. It is easy to see from (1.1) that $[L_3y(t)]' < 0$ for $t \geq t_1$ and hence, we obtain

$$L_2y(t) \geq \int_{t_1}^{t} (L_2y(s))'\mathrm{d}s = \int_{t_1}^{t} \frac{1}{r_2(s)} L_3y(s)\mathrm{d}s \geq R_2(t,t_1)L_3y(t).$$

From this inequality, we get

$$y''(t) \geq \left(\frac{1}{r_1(t)}R_2(t,t_1)\right)^{1/\alpha} L_3^{1/\alpha}y(t).$$

Noting that $L_4y(t) < 0, y(t) > 0$, then there are only the following two possibilities $L_iy(t) > 0, i = 1,2,3$ and $L_1y(t) > 0, L_2y(t) < 0, L_3y(t) > 0$. Thus, $y'(t) > 0$. Now, integrating this inequality twice from t_1 to t and using the fact that L_3y is nonincreasing, we find

$$y'(t) \geq \left[\int_{t_1}^{t} \left(\frac{1}{r_1(s)}R_2(s,t_1)\right)^{1/\alpha} \mathrm{d}s\right] L_3^{1/\alpha}y(t) \quad \text{for } \ t \geq t_1$$

and

$$y(t) \geq \left[\int_{t_1}^{t}\int_{t_1}^{u} \left(\frac{1}{r_1(s)}R_2(s,t_1)\right)^{1/\alpha} \mathrm{d}s\mathrm{d}u\right] L_3^{1/\alpha}y(t) \text{ for } \ t \geq t_1.$$

This completes the proof. □

In the following two lemmas, we consider the second-order delay differential equation

$$(r_2(t)x'(t))') = Q(t)x(h(t)), \tag{2.8}$$

where the function r_2 is as in Eq. (1.1), $h \in C^1(I,R)$ such that $h(t) \leq t$ and $h'(t) \geq 0$ for $t \geq t_0$ and $Q \in C(I,R^+)$.

Lemma 2.3 [17]. *If*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} Q(s)R_2(h(t),h(s))\mathrm{d}s > 1, \tag{2.9}$$

then all bounded solutions of Eq. (2.8) *are oscillatory.*

Lemma 2.4 [17]. *If*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \left((r_2^{-1}(u))\int_{u}^{t} Q(s)\mathrm{d}s\right)\mathrm{d}u > 1, \tag{2.10}$$

then all bounded solutions of Eq. (2.8) *are oscillatory.*

Now, we are ready to establish the main results of this paper.

Theorem 2.1. *Let $\alpha \geq \beta$, conditions* (2.1) *and* (2.2) *hold and Eq.* (2.3) *be nonoscillatory. If there exist two functions ρ and $h \in C^1(I,R)$ such that $g(t) \leq h(t) \leq t, h'(t) \geq 0$ and $\rho(t) > 0$ such that for $t \geq t_0$ such that*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[k\rho(s)q(s) - \frac{A^2(s)}{4B(s)}\right] \mathrm{d}s = \infty \tag{2.11}$$

for any $t_1 \in [t_0, \infty)$*, where*

$$\begin{cases} A(t) = \frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r_1(t)} R_2(t, t_1), \\ B(t) = c^* \rho^{-1}(t) g'(t) (R_{12}^*(g(t), t_1))^{\beta - 1} (R_{12}(g(t), t_1))^{1/\alpha}, \ \ t \geq t_2 \geq t_1 \end{cases} \tag{2.12}$$

and condition (2.9) *or* (2.10) *holds with*

$$Q(t) = [ckg^\beta(t) q(t) R_1(h(t), g(t)) - (p(t)/r_1(t))] \geq 0, \quad t \geq t_1,$$

where c *and* $c^* > 0$ *are any positive constants, then Eq.* (1.1) *is oscillatory.*

Proof. Let $y(t)$ be a nonoscillatory solution of (1.1) on $[t_1, \infty), t \geq t_1$. Without loss of generality, we may assume that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. From Lemma 2.1, it follows that $L_2 y(t) < 0$ or $L_2 y(t) > 0$ for $t \geq t_1$.

If $L_2 y(t) > 0$ for $t \geq t_1$, then one can easily see that $L_3 y(t) > 0$ for $t \geq t_1$. We define

$$w(t) = \rho(t) \frac{L_3 y(t)}{y^\beta(g(t))}, \quad t \geq t_1. \tag{2.13}$$

Differentiating the function w with respect to t and using Eqs. (1.1) and (2.5) in the resulting equation, we have

$$w'(t) \leq -k\rho(t) q(t) + \left[\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r_1(t)} R_2(t, t_1) \right] w(t) - \beta g'(t) \frac{y'(g(t))}{y(g(t))} w(t). \tag{2.14}$$

From (2.6), we get

$$y'(g(t)) = L_1 y(g(t)) \geq R_{12}(g(t), t_1) L_3^{1/\alpha} y(g(t)) \quad \text{for} \ \ t \geq t_1,$$

and

$$\begin{aligned} \frac{y'(g(t))}{y(g(t))} &\geq \left(\frac{R_{12}(g(t), t_1)}{\rho(t)} \right)^{1/\alpha} \frac{\rho^{1/\alpha}(t) L_3^{1/\alpha} y(t)}{y^{\beta/\alpha}(g(t))} y^{\beta/\alpha - 1}(g(t)) \\ &= \left(\frac{R_{12}(g(t), t_1)}{\rho(t)} \right)^{1/\alpha} w^{1/\alpha}(t) y^{\beta/\alpha - 1}(g(t)), \end{aligned}$$

and inequality (2.14) becomes

$$\begin{aligned} w'(t) \leq -k\rho(t) q(t) + \left[\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r_1(t)} R_2(t, t_1) \right] w(t) \\ -\beta g'(t) w^{1+1/\alpha}(t) y^{\beta/\alpha - 1}(g(t)) \left(\frac{R_{12}(g(t), t_1)}{\rho(t)} \right)^{1/\alpha}. \end{aligned} \tag{2.15}$$

Now, there exists a constant c^- and a $t_2 \geq t_1$ such that $L_3 y(t) \leq c^-$ for $t \geq t_2$. It is easy to see that

$$y(t) \leq c_1 \int_{t_2}^{t} \int_{t_2}^{v} \left[\frac{1}{r_1(s)} \int_{t_2}^{s} \frac{1}{r_2(u)} \mathrm{d}u \right]^{1/\alpha} \mathrm{d}s \mathrm{d}v = c_1 R_{12}^*(t, t_2) \tag{2.16}$$

for some constant $c_1 > 0$ and hence we have

$$y^{\beta/\alpha - 1}(g(t)) \geq c_1^{\beta/\alpha - 1} (R_{12}^*(g(t), t_2))^{\beta/\alpha - 1} \quad \text{for} \ \ t \geq t_2. \tag{2.17}$$

From (2.13) and (2.7), we get

$$w(t)=\rho(t)\frac{L_3y(t)}{y^{\beta}(g(t))}\le \rho(t)\frac{L_3y(g(t))}{y^{\beta}(g(t))}$$
$$\le \rho(t)(R_{12}^*(g(t),t_2))^{-\alpha}y^{\alpha-\beta}(g(t)) \quad \text{for } \ t\ge t_1.$$

Using (2.16) in the above inequality, we have

$$w(t)\le (c_1)^{\alpha-\beta}\rho(t)R_{12}^*(g(t),t_1))^{-\beta},$$

and hence

$$w^{1/\alpha-1}(t)\ge (c_1)^{(\alpha-\beta)(1/\alpha-1)}\rho^{1/\alpha-1}(t)R_{12}^*(g(t),t_1))^{-\beta(1/\alpha-1)}. \tag{2.18}$$

Using (2.17) and (2.18) in (2.15), we have

$$w'(t)\le -k\rho(t)q(t)+\left[\tfrac{\rho'(t)}{\rho(t)}-\tfrac{p(t)}{r_1(t)}R_2(t,t_1)\right]w(t)$$
$$-\beta(c_1)^{(\beta-\alpha)}\rho^{-1}(t)g'(t)(R_2^*(g(t),t_2))^{(\beta-1)}R_{12}^*(g(t),t_1))^{1/\alpha}w^2(t),$$

or

$$\begin{aligned}w'(t)&\le -k\rho(t)q(t)+A(t)w(t)-B(t)w^2(t)\\ &=-k\rho(t)q(t)-\left(\sqrt{B(t)}w(t)-\frac{A(t)}{2\sqrt{B(t)}}\right)^2+\frac{A^2(t)}{4B(t)}\\ &=-k\rho(t)q(t)+\frac{A^2(t)}{4B(t)},\end{aligned} \tag{2.19}$$

where $A(t)$ and $B(t)$ are as in (2.12) with $c^*=\beta(c_1)^{(\beta-\alpha)}$.

Integrating inequality (2.19) from t_2 to t, we find

$$\int_{t_2}^{t}\left[k\rho(s)q(s)-\frac{A^2(s)}{4B(s)}\right]\mathrm{d}s\le w(t_2)-w(t)\le w(t_2),$$

which contradicts condition (2.10).

Next, we let $L_2y(t)<0$ for $t\ge t_1$. We consider the function $L_3y(t)$. The case $L_2y(t)\le 0$ cannot hold for all large t, say $t\ge t_2\ge t_1$, since by integration of inequality

$$y'(t)=L_1y(t)\le L_1y(t_2),\quad t\ge t_2,$$

we obtain from (2.1) that $y(t)<0$ for all large t, a contradiction. Thus, we have $y(t)>0, L_1y(t)\ge 0, L_2y(t)<0$ and $L_3y(t)\ge 0$ for all large t, say $t\ge t_3\ge t_2$. From the differential mean value theorem, and combing the monotonicity of y' and $y'(t)>0$, there exists a constant $\theta\in(0,1)$ such that

$$y(t)\ge \theta t y'(t)\quad \text{for } \ t\ge t_3.$$

Using this inequality in Eq. (1.1) we get

$$(r_2(t)(r_1(t)w'(t))^{\alpha})'+p(t)(w'(t))^{\alpha}+k(\theta g(t))^{\beta}q(t)w^{\beta}(g(t))\le 0,$$

where $w(t)=L_1y(t), y''(t)=w'(t)<0$ and so $r_1(t)(w'(t))^{\alpha}<0$ for $t\ge t_3$. Also $L_3y(t)>0$ and so, we have $(r_1(t)(w'(t))^{\alpha})'>0$ for $t\ge t_3$.

Now, for $v \geq u \geq t_3$, we have

$$\begin{aligned} w(u) - w(v) &= -\int_u^v r_1^{-1/\alpha}(\tau)\left(r_1(\tau)(w'(\tau))^\alpha\right)^{1/\alpha} \mathrm{d}\tau \\ &\geq \left(\int_u^v r_1^{-1/\alpha}(\tau)\mathrm{d}\tau\right)(-r_1^{1/\alpha}(v)w'(v)) \\ &= R_1(v,u)(-r_1^{1/\alpha}(v)w'(v)). \end{aligned}$$

Setting $u = g(t)$ and $v = h(t)$, we get

$$w(g(t) \geq R_1(h(t), g(t)))(-r_1^{1/\alpha}(h(t))w'(h(t))) \quad \text{for } \ t \geq t_3,$$

where $z(t) = r_1(t)(-w'(t))^\alpha > 0$ for $t \geq t_3$. From Eq. (1.1) and the fact that x is decreasing and $g(t) \leq h(t) \leq t$, we obtain

$$\begin{aligned} &(r_2(t)z'(t))' + \left(\frac{p(t)}{r_1(t)}\right) z(h(t)) \\ &\qquad \geq k(\theta g^{n-3}(t))^\beta q(t) R_1(h(t), g(t))(z(h(t))(z(h(t))^{\beta/\alpha - 1}. \end{aligned}$$

Since z is decreasing and $\alpha \geq \beta$, there exists a constant $C_1^* > 0$ such that $z^{\beta/\alpha-1}(t) \geq C_1^*$ for $t \geq t_2$. Thus,

$$(r_2(t)z'(t))' \geq \left(C_1^*\theta^\beta k(g^{n-3}(t))^\beta q(t) R_1(h(t), g(t)) - \frac{p(t)}{r_1(t)}\right) z(h(t)).$$

Proceeding exactly as in the proof of Lemmas 2.3 and 2.4, we arrive at the desired conclusion completing the proof of the theorem. □

The following corollary is immediate.

Corollary 2.1. *Let $\alpha \geq \beta$, conditions (2.1), (2.2) hold and Eq. (2.3) be nonoscillatory. If there exist two functions ρ and $h \in C^1(I, R)$ such that $g(t) \leq h(t) \leq t, h'(t) \geq 0$ and $\rho(t) \geq 0$ for $t \geq t_0$ such that the function $A(t) \leq 0$, where $A(t)$ is defined as in (2.12),*

$$\limsup_{t\to\infty} \int_{t_1}^\infty \rho(s)q(s)\mathrm{d}s = \infty \tag{2.20}$$

for any $t_1 \in [t_0, \infty)$ and condition (2.9) or (2.10) holds with $Q(t)$ is as Theorem 2.1, then Eq. (1.1) is oscillatory.

The following examples are illustrative.

Example 2.1. Consider the equation

$$((y''(t))^3)'' + 9(y''(t))^3 + 6y(t - 2\pi) = 0. \tag{2.21}$$

It is easy to check that all conditions of Corollary 2.1 are satisfied and hence Eq. (2.21) is oscillatory. One such solution is $y(t) = \sin t$.

Example 2.2. Consider the equation

$$((y''(t))^3)'' + (y''(t))^3 + \frac{10}{\mathrm{e}^9} y^3(t-1) = 0. \tag{2.22}$$

Here, we take $k = 1, \rho(t) = 1$ and $h(t) = t - 1/2$. Now, it is easy to check that all hypotheses of Theorem 2.1 are fulfilled except that $Q(t)$ is negative. We note that Eq. (2.22) admits the nonoscillatory solution $y(t) = \mathrm{e}^{-t}$.

For $t \geq t_1 \geq t_0$, we let

$$P(t) = \frac{p(t)}{r_1(t)} R_2(t, t_1), \quad Q^-(t) = kq(t)(R_{12}^*(g(t), t_1))^\beta \quad \text{and} \quad \mu(t) = \exp\left(\int_{t_1}^t P(s)\mathrm{d}s\right).$$

Now, we present the following comparison result.

Theorem 2.2. *Let $\alpha \geq \beta$. Assume that conditions (2.1) and (2.2) hold, Eq. (2.3) is nonoscillatory and there exists a function $h \in C^1(I, R)$ such that $g(t) \leq h(t) \leq t, h'(t) \geq 0$ for $t \geq t_0$ and condition (2.9) or (2.10) holds with $Q(t)$ is as Theorem 2.1. If every solution of the first-order delay equation*

$$z'(t) + (\mu(g(t)))^{1+\beta/\alpha} Q^-(t) z^{\beta/\alpha}(g(t)) = 0 \tag{2.23}$$

is oscillatory, then Eq. (1.1) is oscillatory.

Proof. Let $y(t)$ be a nonoscillatory solution of (1.1) on $[t_1, \infty), t \geq t_1$. Without loss of generality, we may assume that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. From Lemma 2.1, it follows that $L_2 y(t) < 0$ or $L_2 y(t) > 0$ for $t \geq t_1$. If $L_2 y(t) > 0$ for $t \geq t_1$, then one can easily see that $L_3 y(t) > 0$ for $t \geq t_1$. There exists a $t_2 \geq t_1$ such that $g(t) \geq t_1$ for $t \geq t_2$ and

$$y(g(t)) \geq R_{12}^*(g(t), t_1) L_3^{1/\alpha} y(g(t)) \quad \text{for} \quad t \geq t_2. \tag{2.24}$$

Using (2.5) and (2.24) in Eq. (1.1), we have

$$(L_3 y(t))' + \left(\frac{p(t)}{r_1(t)}\right) R_2(t, t_1) L_3 y(t) + kq(t)(R_{12}^*(g(t), t_1))^\beta (L_3 y(g(t)))^{\beta/\alpha} \leq 0 \quad \text{for} \quad t \geq t_2$$

or

$$w'(t) + P(t) w(t) + Q^-(t) w^{\beta/\alpha}(t) \leq 0 \quad \text{for} \quad t \geq t_2,$$

where $w(t) = L_2 y(t)$ or

$$(\mu(t) w(t))' + \mu(t) Q^-(t) w^{\beta/\alpha}(t) \leq 0 \quad \text{for} \quad t \geq t_2.$$

Setting $z(t) = \mu(t) w(t)$ in the above inequality and noting that $\mu(g(t)) \leq \mu(t)$, we obtain

$$z'(t) + (\mu(g(t)))^{1+\beta/\alpha} Q^-(t) z^{\beta/\alpha}(g(t)) \leq 0.$$

This inequality has a positive solution and by [1, Corollary 2.3.5], we see that Eq. (2.23) has a positive solution, a contradiction. The case is similar to that of Theorem 2.1 and hence is omitted. This completes the proof. □

The following corollary is immediate.

Corollary 2.2. *Let $\alpha \geq \beta$, conditions (2.1) and (2.2) hold and equation (2.3) be nonoscillatory and there exists a function $h \in C^1(I, R)$ such that $g(t) \leq$*

$h(t) \le t$ and $h'(t) \ge 0$ for $t \ge t_0$ and condition (2.9) or (2.10) hold with $Q(t)$ being as in Theorem 2.1. If

$$\begin{cases} \liminf\limits_{t\to\infty} \int_{g(t)}^{t} \mu^2(g(s))Q^-(s)ds > 1/e & \text{when } \alpha = \beta, \\ \int^{\infty} \mu^{1+\beta/\alpha}(g(s))Q^-(s)ds = \infty & \text{when } \alpha > \beta \end{cases} \tag{2.25}$$

then Eq. (1.1) is oscillatory.

Next, if Eq. (2.3) is oscillatory, we give the following result.

Theorem 2.3. *Let conditions (2.1) and (2.2) hold and Eq. (2.3) be oscillatory. If there exists a function $h \in C(I, R)$ such that $g(t) \le h(t) \le t$ and $h'(t) \ge 0$ for $t \ge t_0$ such that condition (2.9) or (2.10) holds with $Q(t)$ being as in Theorem 2.1, then every solution $y(t)$ of (1.1) either $y(t)$ is oscillatory or $y'(t)$ is oscillatory.*

Proof. Let $y(t)$ be a nonoscillatory solution of (1.1) on $[t_1, \infty), t \ge t_1$. Without loss of generality, we may assume that $y(t) > 0$ and $y(g(t)) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. Now, we consider the cases $L_2y(t) < 0$ or $L_2y(t) > 0$ for $t \ge t_1$. If $L_2y(t) > 0$ for $t \ge t_1$ holds, then Eq. (1.1) becomes

$$(r_2(t)x'(t))' + \frac{p(t)}{r_1(t)}x(t) \le 0 \quad \text{for} \ \ t \ge t_2 \ge t_1,$$

where $x(t) = L_2y(t)$. By [12, Lemma 2.6], Eq. (2.3) has a positive solution, a contradiction. The proof of the case when $L_2y(t) < 0$ for $t \ge t_2 \ge t_1$ is similar to that of Theorem 2.1 and hence is omitted. This completes the proof of the theorem. □

As an illustrative example, we consider the equation

$$y^{(4)}(t) + \frac{1}{2}y^{(2)}(t) + \frac{1}{2}y(t-\pi) = 0. \tag{2.26}$$

Here, $\alpha = \beta = 1$ and let $h(t) = t - \pi$. It is easy to check that all the hypotheses of Theorem 2.2 are satisfied and hence every solution y of Eq. (2.26) is oscillatory or y' is oscillatory. One such solution is $y(t) = \sin t$. We note that none of the results in [2,7,9–14] are applicable to Eq. (2.26).

Finally, we can easily extend Theorem 2.3 to the equation

$$(r_2(t)(r_1(t)(y'(t))^{\alpha})')' + p(t)y'(h(t)) + q(t)f(y(g(t))) = 0, \tag{2.27}$$

where $h \in C(I, R)$ such that $g(t) \le h(t) \le t$ and $h'(t) \ge 0$ for $t \ge t_0$.

Theorem 2.4. *Let conditions (2.1) and (2.2) hold and the equation*

$$(r_2(t)x'(t)' + \frac{p(t)}{r_1(h(t))}x(h(t)) = 0 \tag{2.28}$$

be oscillatory. If condition (2.9) or (2.10) holds with

$$Q(t) = [ckq(t)R_1(h(t), g(t)) - (p(t)/r_1(h(t))] \ge 0 \quad \textit{for} \ \ t \ge t_1,$$

where c is any positive constant, then every solution y of Eq. (2.27) either $y(t)$ is oscillatory or $y'(t)$ is oscillatory.

Proof. Let $y(t)$ be a nonoscillatory solution of (2.27) on $[t_1, \infty), t \geq t_1$. Without loss of generality, we may assume that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. As in the proof of Theorem 2.2 we obtain either $L_2 y(t) < 0$ or $L_2 y(t) > 0$ for $t \geq t_1$. If $L_2 y(t) > 0$ for $t \geq t_1$ holds, then Eq. (2.27) becomes

$$(r_2(t)x'(t)' + \frac{p(t)}{r_1(h(t))} x(h(t)) \leq 0 \quad \text{for} \quad t \geq t_2 \geq t_1,$$

where $x(t) = L_2 y(t) > 0$. By [12, Lemma 2.6], Eq. (2.28) has a positive solution, a contradiction. The proof of the case when $L_2 y(t) < 0$ for $t \geq t_2 \geq t_1$ is similar to that of Theorem 2.1 and hence is omitted. This completes the proof of the theorem. □

We note that there are many criteria in the literature for the oscillation of second-order dynamic equations, and so by applying these results to Eqs. (1.1) and (2.27), we can obtain many oscillation results, more, for example, than those presented in [1,6].

The following examples are illustrative.

Example 2.3. Consider the equation

$$y^{(4)}(t) + y^{(2)}(t - \pi) + 2y(t - 2\pi) = 0. \tag{2.29}$$

It is easy to check that all the hypotheses of Theorem 2.4 are satisfied with $\alpha = \beta = 1$ and hence every solution $y(t)$ of Eq. (2.29) either $y(t)$ is oscillatory or $y'(t)$ is oscillatory. One such solution is $y(t) = \sin t$.

We note that none of the known results appeared in the literature are applicable to this equation because of the delay the appeared in the damping term.

Next, we establish new oscillation results for Eq. (1.1) using the integral averaging technique due to Philos [16]. We need the class of function $\mathcal{H}$. Let

$$\mathbb{D}_0 = \{(t, s) : t > s > t_0\} \quad \text{and} \quad \mathbb{D} = \{(t, s) : t \leq s > t_0\}.$$

A function $H \in C(\mathbb{D}, \mathbb{R})$ is said to be the class $\mathcal{H}$ if

(i) $H(t, s) > 0 \quad$ for all $(t, s) \in \mathbb{D}_0, H(t, t) = 0$;
(ii) H has a continuous and nonpositive partial derivatives on $\mathbb{D}_0$ with respect to the second variable and for a positive continuous function $\overline{h}(t, s)$ such that

$$\frac{\partial H(t, s)}{\partial s} = -\bar{h}(t, s)\sqrt{H(t, s)} \quad \text{for all} \quad (t, s) \in \mathbb{D}_0.$$

For the choice $H(t, s) = (t - s)^n (n \geq 1)$, the Philos type conditions reduce to the Kamener type ones.

Theorem 2.5. *Let $\alpha > 1$, conditions (2.1) and (2.2) hold and the Eq. (2.3) be nonoscillatory. If there exist two functions g and $h \in C^1(I, R)$ such that $g(t) \leq h(t) \leq t$ and $h'(t) \geq 0$ and $g(t) > 0$ for $t \geq t_0$ and $H \in \mathcal{H}$ such that*

$$\limsup_{t \to \infty} \frac{1}{H(t, t_1)} \int_{t_1}^{t} \left[k g(s) H(t, s) q(s) - \frac{P^2(t, s)}{4B(s)} \right] \mathrm{d}s = \infty, \tag{2.30}$$

for all large $t \geq t_1$, where

$$P(t,s) = \bar{h}(t,s) - \sqrt{H(t,s)}\left[\frac{g'(s)}{g(s)} - \rho(s)\frac{R_2(t,t_1)}{r_1(s)}\right],$$

$B(s)$ is defined as in Theorem 2.1, and condition (2.9) or (2.10) holds with Q as in Theorem 2.1, then Eq. (1.1) is oscillatory.

Proof. Let $y(t)$ be a nonoscillatory solution of Eq. (1.1), say $y(t) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Proceeding as in the proof of Theorem 2.1, we obtain the inequality (2.19), i.e.,

$$w'(t) \leq -kg(t)q(t) + A(t)w(t) - B(t)w^2(t),$$

and so,

$$\begin{aligned}
\int_{t_1}^{t} kH(t,s)g(s)q(s)\mathrm{d}s &\leq \int_{t_1}^{t} H(t,s)[-w'(s) + A(s)w(s) - B(s)w^2(s)]\mathrm{d}s \\
&= -H(t,s)w(s)|_{t_1}^{t} + \int_{t_1}^{t}\left[\frac{\partial H(t,s)}{\partial s}w(s) + H(t,s)(A(s)w(s) - B(s)w^2(s))\right]\mathrm{d}s \\
&= H(t,t_1)w(t_1) - \int_{t_1}^{t}\left[w^2(s)B(s)H(t,s) + w(s)\left[\bar{h}(t,s)\sqrt{H(t,s)}\right.\right. \\
&\quad \left.\left. - H(t,s)A(s)\right]\right]\mathrm{d}s \leq H(t,t_1)w(t_1) + \int_{t_1}^{t}\frac{P^2(t,s)}{4B(s)}\mathrm{d}s.
\end{aligned}$$

Thus, we obtain

$$\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[kg(s)H(t,s)q(s) - \frac{P^2(t,s)}{4B(s)}\right]\mathrm{d}s \leq w(t_1),$$

which contradicts condition (2.30). The rest of the proof is similar to that of Theorem 2.1 and hence is omitted. □

Theorem 2.6. *Let the hypotheses of Theorem 2.2 hold. Moreover, suppose that for ever $t_1 > t_0$,*

$$0 < \inf_{s \geq t_1}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_1)}\right] < \infty, \tag{2.31}$$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\frac{g(s)r_1(h(s))P^2(t,s)}{R_2(s,t_1)g'(s)}\mathrm{d}s < \infty,$$

and there exists $\psi \in C[t_0,\infty)$ such that

$$\int_{t_1}^{t}\psi_+^2(s)\frac{R_2(s,t_1)g'(s)}{g(s)r_1(h(s))}, \quad \psi_+ = \max\{\psi, 0\},$$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[kg(s)H(t,s)q(s) - \frac{P^2(t,s)}{4B(s)}\right]\mathrm{d}s \geq \psi(t_1). \tag{2.32}$$

Then, Eq. (1.1) is oscillatory.

Proof. Let $y(t)$ be a nonoscillatory solution of Eq. (1.1), say $y(t) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Proceeding as in the proof of Theorem 2.2, we have

$$\int_{t_1}^{t} kH(t,s)g(s)q(s) \leq H(t,t_1)w(t_1) + \int_{t_1}^{t} \frac{P^2(t,s)}{4B(s)}\mathrm{d}s - \int_{t_1}^{t} \left[\sqrt{H(t,s)B(s)}w(s) + \frac{P(t,s)}{4B(s)}\right]^2 \mathrm{d}s.$$

Then,

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)} \left[\int_{t_1}^{t} kH(t,s)g(s)q(s) - \frac{P^2(t,s)}{4B(s)}\right] \mathrm{d}s \leq w(t_1) - \liminf_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[\sqrt{H(t,s)B(s)}w(s) + \frac{P(t,s)}{2\sqrt{B(s)}}\right]^2 \mathrm{d}s.$$

Using (2.32), we obtain

$$w(t_1) \geq \psi(t_1) + \liminf_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[\sqrt{H(t,s)B(s)}w(s) + \frac{P(t,s)}{2\sqrt{B(s)}}\right]^2 \mathrm{d}s,$$

and hence

$$\liminf_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[\sqrt{H(t,s)B(s)}w(s) + \frac{P(t,s)}{2\sqrt{B(s)}}\right]^2 \mathrm{d}s < \infty. \tag{2.33}$$

Define

$$c_1 = \frac{1}{H(t,t_1)} \int_{t_1}^{t} H(t,s)B(s)w^2(s)\mathrm{d}s, \;\; c_2 = \frac{1}{H(t,t_1)} \int_{t_1}^{t} \sqrt{H(t,s)}P(t,s)w(s)\mathrm{d}s.$$

It follows from (2.33) that

$$\liminf_{t\to\infty} [c_1(t) + c_2(t)] < \infty.$$

The remainder of the proof is similar to that of Theorem 3 in [18] and hence is omitted. The rest of the proof of the case if $y(t) > 0$ and $L_1 y(t) < 0$ is similar to that of Theorem 2.1 and hence is omitted. □

3. General Remarks

1. The results of this paper are presented in a form that is essentially new and of a high degree of generality.
2. It would be of interest to consider Eqs. (1.1) and (2.27) and try to obtain some oscillation criteria if for $p(t) < 0$ and $q(t) < 0$.
3. Finally, we note that our oscillation results are applicable to Eq. (1.1) if $g(t) < t$. Thus, as is well known, it is the delay in Eq. (1.1) that can generate the oscillations.
4. The results of this paper can be easily extended to dynamic equations of the form

$$(r_2(t)(r_1(t)(y^{\Delta\Delta}(t))^\alpha)')' + p(t)(y^{\Delta\Delta}(t))^\alpha + q(t)f(y(g(t))) = 0,$$

where r_1, r_2, p, q and g are rd-continuous functions defined on any time scale $\mathbb{T}$ with $\sup \mathbb{T} = \infty$. The function f and the constant α are as in Eq. (1.1). The details are left to the reader.

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