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ON VOLTERRA-INTEGRO DYNAMICAL SYSTEMS ON TIME SCALES

ELVAN AKIN[1] AND ÖZKAN ÖZTÜRK[2]

[1,2]Department of Mathematics and Statistics
Missouri University of Science and Technology
Rolla, MO, 65409-0020, USA

ABSTRACT: In this paper, we classify nonoscillatory solutions of a two dimensional nonlinear Volterra-integro time scale system and also provide sufficient conditions for the existence of such solutions via the Knaster fixed point theorem.

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1. INTRODUCTION

The Volterra-integro equations showed up after its foundation by Volterra and then it has become very popular to be used in many physical applications such as glass-forming process, nanohydrodynamics, heat transfer, diffusion process in general, neutron diffusion and biological species coexisting together with increasing and decreasing rates of generating, and wind ripple in the desert. Recently, it has been investigating the stability, instability and some numerical approximations to solutions of Volterra integral equations, see [1, 2, 3, 4, 5, 6, 7, 8] for more details.

This paper deals with systems of the form

$$\begin{cases} x^{\Delta}(t) = a(t)x(t) + \displaystyle\int_{t_0}^{t} p(t,s)f(y(s))\Delta s \\ y^{\Delta}(t) = -b(t)y^{\sigma}(t) - \displaystyle\int_{t_0}^{t} q(t,s)g(x(s))\Delta s \end{cases} \tag{1}$$

and

$$\begin{cases} x^{\Delta}(t) = a(t)x(t) - \displaystyle\int_{t_0}^{t} p(t,s)f(y(s))\Delta s \\ y^{\Delta}(t) = b(t)y(t) - \displaystyle\int_{t_0}^{t} q(t,s)g(x(s))\Delta s, \end{cases} \tag{2}$$

where $f, g \in C(\mathbb{R}, \mathbb{R})$ are nondecreasing such that $uf(u) > 0$, $ug(u) > 0$ for $u \neq 0$, $a, b \in \mathrm{C_{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$ and $p, q \in C([t_0, \infty)_{\mathbb{T}} \times [t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$. A time scale, denoted by $\mathbb{T}$, is a closed subset of real numbers. For time scales calculus we refer readers to books [9] and [10]. Throughout this paper, we assume that $\mathbb{T}$ is unbounded above and whenever we write $t \geq t_0$, we mean that $t \in [t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$.

We call (x, y) a *proper solution* of (1) (or (2)) if $\sup\{|x(s)|, |y(s)| : s \in [t, \infty)_{\mathbb{T}}\} > 0$ for $t \geq t_0$ and it is defined on $[t_0, \infty)_{\mathbb{T}}$. A solution (x, y) of (1) (or (2)) is said to be *nonoscillatory* if the component functions x and y are both nonoscillatory, i.e., either eventually positive or eventually negative. Otherwise, it is said to be *oscillatory*. One can observe that if x is nonoscillatory, then y has to be nonoscillatory. By a *positive solution*, we mean that x and y have the same sign while by a *negative solution* we mean that x and y have the different sign.

In [11], the authors only consider positive solutions of the following system

$$\begin{cases} x^{\Delta}(t) = a(t)x(t) + \displaystyle\int_{t_0}^{t} p(t,s)f(y(s))\Delta s \\ y^{\Delta}(t) = b(t)y(t) + \displaystyle\int_{t_0}^{t} q(t,s)g(x(s))\Delta s. \end{cases} \tag{3}$$

By the method of sign of solutions, the existence of negative solutions of system (3) can not be obtained. Therefore, we urge to consider system (2) for the existence of negative solutions in Section 2. One can also consider system (1) in which the forward jump operator σ is included. Observe that the existence of positive solutions of (1) can be investigated, see Section 3. In both subsections, without loss of generality we assume that the first component of such solutions is always positive.

To reach our goal, we need the following preliminary results. If $\sup \mathbb{T} < \infty$, then $\mathbb{T}^{\kappa} = \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}]$, and $\mathbb{T}^{\kappa} = \mathbb{T}$ if $\sup \mathbb{T} = \infty$. We say that $h_1 : \mathbb{T} \to \mathbb{T}$ is regressive, denoted by $\mathcal{R}$, provided $1 + h_1(t)\mu(t) \neq 0$ for $t \in \mathbb{T}^{\kappa}$. If $h_1 \in \mathcal{R}$, then the first order linear dynamic equation $z^{\Delta} = h_1(t)z$ is called *regressive.* The exponential function on time scales $e_{h_1}(\cdot, t_0)$ is a solution of the initial value problem $z^{\Delta} = h_1(t)z, \ z(t_0) = 1$ and it is known that if $h_1 \in \mathcal{R}$ and $1 + \mu h_1 > 0$ on $\mathbb{T}^{\kappa}$, then $e_{h_1}(t, t_0) > 0$ for all $t \in \mathbb{T}$, see [9, Theorem 2.44]. Also if $h_1 \geq 0$, then $e_{h_1}(\cdot, t_0) \geq 1$, see [12, Remark 2.12]. More properties of exponential functions on time scales can be found in [9, Theorem 2.36].

Lemma 1. *[9, Theorem 2.77] Suppose that $z^{\Delta} = h_1 z + h_2$ is regressive. Let $t_0 \in \mathbb{T}$ and $z_0 \in \mathbb{R}$. Then the unique solution of the initial value problem $z^{\Delta}(t) = h_1(t)z + h_2(t), \ z(t_0) = z_0$ is given by*

$$z(t) = e_{h_1}(t, t_0)z_0 + \int_{t_0}^{t} e_{h_1}(t, \sigma(\tau))h_2(\tau)\Delta\tau.$$

Next, we provide the Knaster fixed point theorem in order to show the existence of nonoscillatory solutions of system (1) and (2), see [13].

Theorem 1 (Knaster Fixed Point Theorem)**.** *If $(M, \leq)$ is a complete lattice and $T : M \to M$ is order-preserving (also called monotone or isotone), then T has a fixed point. In fact, the set of fixed points of T is a complete lattice.*

2. MAIN RESULTS

2.1. EXISTENCE OF POSITIVE SOLUTIONS OF SYSTEM (1)

Let M^+ be the set of all positive solutions of system (1). Since the first and second equations of system (1) give us that x and y are eventually increasing and decreasing, respectively, we have the following subclasses for $0 < c, d < \infty$:

$$M^+_{B,B} = \left\{(x,y) \in M^+ : \lim_{t\to\infty} x(t) = c, \quad \lim_{t\to\infty} y(t) = d\right\},$$
$$M^+_{B,0} = \left\{(x,y) \in M^+ : \lim_{t\to\infty} x(t) = c, \quad \lim_{t\to\infty} y(t) = 0\right\},$$
$$M^+_{\infty,B} = \left\{(x,y) \in M^+ : \lim_{t\to\infty} x(t) = \infty, \lim_{t\to\infty} y(t) = d\right\},$$
$$M^+_{\infty,0} = \left\{(x,y) \in M^+ : \lim_{t\to\infty} x(t) = \infty, \lim_{t\to\infty} y(t) = 0\right\}.$$

To accomplish the existence of positive solutions in subclasses given above by using the Knaster fixed point theorem, we set

$$A(t) = \int_t^\infty a(s)\Delta s, \quad B(t) = \int_t^\infty b(s)\Delta s,$$
$$Y_1 = \int_{t_0}^\infty \left[\int_{t_0}^t q(t,s)g(r(s))\Delta s\right]\Delta t,$$
$$Y_2 = \int_{t_0}^\infty \left[\int_{t_0}^t p(t,s)f\left(k\int_s^\infty \left(\int_{t_0}^u q(u,v)\Delta v\right)\Delta u\right)\Delta s\right]\Delta t,$$

where $k > 0$ and $r(\cdot)$ is defined for some $c > 0$ and $x_0 \geq 0$. Throughout this paper, we assume $A(t_0) < \infty$ and $B(t_0) < \infty$.

Theorem 2. *If $Y_1 < \infty$ and $Y_2 = \infty$, then $M^+_{\infty,0} \neq \emptyset$.*

Proof. Suppose $Y_1 < \infty$, $Y_2 = \infty$ and $B(t_0) < \infty$. Then we can choose $t_1 \geq t_0$ and $d_1 > 0$ such that $Y_1 < d_1$ and $B(t_1) < \frac{1}{2}$. Let X be the Banach space of all continuous real valued functions endowed with the norm $\|y\| = \sup_{t\in[t_1,\infty)_\mathbb{T}} |y(t)|$ and with usual pointwise ordering $\leq$. Define a subset Ω of X as

$$\Omega := \{y \in X : \quad d_1 \int_t^\infty \left(\int_{t_0}^u q(u,s)\Delta s\right)\Delta u \leq y(t) \leq 2d_1, \quad t \geq t_1\}.$$

For any subset B of Ω, it is clear that $\inf B \in \Omega$ and $\sup B \in \Omega$. Let us define an operator $F : \Omega \to X$ as

$$(Fy)(t) = \int_t^\infty b(s)y^\sigma(s)\Delta s$$

$$+\int_t^\infty \int_{t_0}^u q(u,s) g\Bigg(e_a(s,t_0)x_0$$
$$+\int_{t_0}^s e_a(s,\sigma(\tau)) \left(\int_{t_0}^\tau p(\tau,v) f(y(v))\Delta v \right) \Bigg) \Delta s$$

Let us show that F is an increasing mapping into itself. Indeed, for $x_1 \le x_2$ we have $Fx_1 \le Fx_2$, where $x_1, x_2 \in \Omega$, i.e. F is an increasing mapping. Next, we show $F : \Omega \to \Omega$.. Since $e_a(\cdot, t_0) > 1$, $Y_1 < 2d_1$, and $Y_2 = \infty$

$$\begin{aligned}
&g(1) \int_t^\infty \int_{t_0}^u q(u,s)\Delta s \le (Fy)(t) \\
&\le d_1 + \int_t^\infty \int_{t_0}^u q(u,s) g \left(e_a(s,t_0)x_0 + \int_{t_0}^s e_a(s,\sigma(\tau)) \left(\int_{t_0}^\tau p(\tau,v) f(2d_1)\Delta v \right) \right) \Delta s \\
&\le 2d_1
\end{aligned}$$

for $t \ge t_1$, which implies $F : \Omega \to \Omega$. Then by Theorem 1, there exists $\bar{y} \in \Omega$ such that $\bar{y} = F\bar{y}$. By taking the derivative of $\bar{y}$, we have

$$\begin{aligned}
&\bar{y}^\Delta(t) = -b(t)\bar{y}^\sigma(t) \\
&- \int_{t_0}^t q(t,s) g \left(e_a(s,t_0)x_0 + \int_{t_0}^s e_a(s,\sigma(\tau)) \left(\int_{t_0}^\tau p(\tau,v) f(\bar{y}(v))\Delta v \right) \Delta\tau \right) \Delta s
\end{aligned}$$

for $t \ge t_1$. Setting

$$\bar{x}(t) = e_a(t,t_0)\bar{x}_0 + \int_{t_0}^t e_a(t,\sigma(\tau)) \left(\int_{t_0}^\tau p(\tau,v) f(\bar{y}(v))\Delta v \right) \Delta\tau \tag{4}$$

implies

$$\bar{x}^\Delta(t) = a(t)\bar{x}(t) + \int_{t_0}^t p(t,s) f(\bar{y}(s))\Delta s,$$

by Lemma 1, i.e., $(\bar{x}, \bar{y})$ is a nonoscillatory solution of system (1). Note also that $\bar{x}(t) > 0$ and $\bar{y}(t) > 0$ for $t \ge t_1$ because $e(\cdot, t_0) > 0$ and $\bar{x}(t) \to \infty$ and $\bar{y}(t) \to 0$ as $t \to \infty$, i.e., $M^+_{\infty,0} \ne \emptyset$. ☐

The following theorem can be proven similar to Theorem 2 by setting an operator $Ty = d + Fy$, where $d > 0$ and Fy is defined as in the proof of Theorem 2.

Theorem 3. *If $Y_1 < \infty$ and $Y_2 = \infty$, then $M^+_{\infty,B} \neq \emptyset$.*

2.2. EXISTENCE OF NEGATIVE SOLUTIONS OF SYSTEM (2)

Let M^- be the set of all negative solutions of system (2). Then by the similar discussion as in Subsection 2.1, following subclasses are obtained for $0 < c < \infty$ and $-\infty < d < 0$.

$$\begin{aligned}
M^-_{B,B} &= \left\{(x,y) \in M^- : \lim_{t\to\infty} x(t) = c, \quad \lim_{t\to\infty} y(t) = d\right\}, \\
M^-_{B,\infty} &= \left\{(x,y) \in M^- : \lim_{t\to\infty} x(t) = c, \quad \lim_{t\to\infty} y(t) = -\infty\right\}, \\
M^-_{\infty,B} &= \left\{(x,y) \in M^- : \lim_{t\to\infty} x(t) = \infty, \lim_{t\to\infty} y(t) = d\right\}, \\
M^-_{\infty,\infty} &= \left\{(x,y) \in M^- : \lim_{t\to\infty} x(t) = \infty, \lim_{t\to\infty} y(t) = -\infty\right\}.
\end{aligned}$$

For the convenience, set

$$\begin{aligned}
Y_3 &= \int_{t_0}^{\infty} \left[\int_{t_0}^{t} p(t,s) f(h(s)) \Delta s\right] \Delta t, \\
Y_4 &= \int_{t_0}^{\infty} \left[\int_{t_0}^{t} q(t,s) \Delta s\right] \Delta t,
\end{aligned}$$

where $h(\cdot)$ is defined for some constant l and $y_0 = 0$.

Theorem 4. *If $Y_3 < \infty$ and $Y_4 = \infty$, then $M^-_{B,\infty} \neq \emptyset$.*

Proof. Suppose that $Y_3 < \infty$ and $Y_4 = \infty$. Then there exist $c_1 > 0$ and $t_1 \geq t_0$ such that

$$\int_{t_1}^{\infty} \left[\int_{t_1}^{t} p(t,s) f(h(s)) \Delta s\right] \Delta t < \frac{c_1}{4} \tag{5}$$

and $A(t_1) < \frac{1}{4}$. Let X be the Banach space of all continuous real valued functions endowed with the norm $\|x\| = \sup_{t\in[t_1,\infty)_{\mathbb{T}}} |x(t)|$ and with usual pointwise ordering $\leq$. Define a subset Ω of X as

$$\Omega := \{x \in X : \quad \frac{c_1}{2} \leq x(t) \leq c_1, \quad t \geq t_1\}.$$

For any subset B of Ω, it is clear that $\inf B \in \Omega$ and $\sup B \in \Omega$. Let us define an operator $F : \Omega \to X$ as

$$\begin{aligned}(Fx)(t) &= \frac{c_1}{2} + \int_{t_1}^{t} a(s)x(s)\Delta s \\ &- \int_{t_1}^{t}\int_{t_1}^{u} p(u,s) f\left(-\int_{t_1}^{s} e_b(s,\sigma(\tau))\left(\int_{t_1}^{\tau} q(\tau,v)g(x(v))\Delta v\right)\Delta\tau\right)\Delta s\end{aligned}$$

It could be shown that F is an increasing mapping and it is clear that $(Fx)(t) > \frac{c_1}{2}$ for $t \geq t_1$. Also,

$$(Fx)(t) \leq \frac{c_1}{2} + \frac{c_1}{4} + \frac{c_1}{4} = c_1, \quad i.e.,\ F : \Omega \to \Omega.$$

Then by Theorem 1, there exists an $\bar{x} \in \Omega$ such that $\bar{x} = F\bar{x}$ and $\bar{x} > 0$ eventually. So as $t \to \infty$, we have $\bar{x}(t) \to \alpha$, where $0 < \alpha < \infty$. Taking the derivative of $\bar{x}$ yields

$$\begin{aligned}\bar{x}^{\Delta}(t) &= a(t)\bar{x}(t) \\ &- \int_{t_1}^{t} p(t,s) f\left(-\int_{t_1}^{s} e_b(s,\sigma(\tau))\left(\int_{t_1}^{\tau} q(\tau,v)g(\bar{x}(v))\Delta v\right)\Delta\tau\right)\Delta s > 0\end{aligned}$$

for $t \geq t_1$. Setting $\bar{y}(t) = -\int_{t_1}^{t} e_b(t,\sigma(\tau))\left(\int_{t_1}^{\tau} q(\tau,v)g(\bar{x}(v))\Delta v\right)\Delta\tau < 0$ and taking the derivative of the latter equation give us $(\bar{x},\bar{y})$ is a nonoscillatory solution of system (2) by Lemma 1. In addition, one can have

$$\bar{y}(t) < -k_2 \int_{t_1}^{t}\left(\int_{t_1}^{\tau} q(\tau,v)\Delta v\right)\Delta\tau, \quad \text{where} \quad k_2 = g(\frac{c_1}{2}) > 0.$$

since $e_b(\cdot,t_0) > 1$. So as $t \to \infty$, it follows $\bar{y}(t) \to -\infty$, i.e., $M^{-}_{B,\infty} \neq \emptyset$.

$\square$

Theorem 5. *Any nonoscillatory solution in M^- belongs to $M^-_{\infty,\infty}$ if*

$$\int_{t_0}^{\infty}\left[\int_{t_0}^{t} p(t,s)\Delta s\right]\Delta t=\infty \quad \textit{and} \quad Y_4=\infty. \tag{6}$$

Proof. Suppose that (x,y) is a nonoscillatory solution of system (2) in M^- and (6) hold. Then system (2) gives us x is increasing and y is decreasing eventually. Then there exist $t_1 \geq t_0$, $k_1 < 0$ and $k_2 > 0$ such that $f(y(t)) \leq k_1$ and $g(x(t)) \geq k_2$ for $t \geq t_1$. By integrating the first and second equations of system (2), we have

$$\begin{aligned} x(t) &\geq -k_1 \int_{t_1}^{t}\left[\int_{t_0}^{u} p(u,s)\Delta s\right]\Delta u \\ y(t) &\leq -k_2 \int_{t_1}^{t}\left[\int_{t_0}^{u} q(u,s)\Delta s\right]\Delta u, \end{aligned}$$

respectively. Then $(x,y) \in M^-_{\infty,\infty}$. □

3. CONCLUSION

Note that the existence in $M^+_{B,B}$ and $M^+_{B,0}$ (bounded solutions in M^+) is not obtained in Subsection 2.1 for general time scales. The main reason for this is as follows: When the operator is chosen depends on x, the component function y cannot be positive, which is a contradiction to the fact $x > 0$ eventually. Therefore, the operators in Subsection 2.1 must depend on y. Once the operators depend on y, limit of the component function y cannot be bounded due to the fact that exponential function is unbounded above. In addition, the results for system (1) can be obtained without σ in system (1). Similarly, in Subsection 2.2, the nonemptiness of $M^-_{\infty,B}$ and $M^-_{B,B}$ are not acquired because of the exponential function. Observe also that system (2) is considered without σ since y cannot be solved explicitly.

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