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THE UPPER AND LOWER SOLUTION METHOD FOR DIFFERENTIAL INCLUSIONS VIA A LATTICE FIXED POINT THEOREM

RAVI P. AGARWAL, B.C. DHAGE, AND DONAL O'REGAN

Department of Mathematical Science, Florida Institute of Technology, Melbourne, Florida 32901, USA.
Kasubai, Gurukul Colony, Almedpur 413515, Maharashtra, India.
Department of Mathematics, National University of Ireland, Galway, Ireland.

ABSTRACT. In this paper the existence of extremal solutions to differential inclusions is obtained by combining the method of upper and lower solutions with a lattice fixed point theorem.

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1. PRELIMINARIES

Let $\mathbb{R}$ denote the real line and $2^{\mathbb{R}}$ the family of nonempty subsets of $\mathbb{R}$. We let $I = [0, a]$ (here $a > 0$ is fixed) and we will consider in this paper the initial value problem

$$(1.1) \qquad \begin{cases} x^{(n)} \in F(t, x) \quad \text{a.e.} \quad t \in I \\ x^{(i)}(0) = x_i \in \mathbb{R}, \quad i = 0, 1, \ldots, n-1, \end{cases}$$

with $F : I \times \mathbb{R} \to 2^{\mathbb{R}}$ a multivalued map and $n \in \mathbb{N} = \{1, 2, \ldots\}$ fixed. By a solution to (1.1) we mean a function $x \in AC^n(I)$ that satisfies (1.1); here $AC^n(I)$ denotes the class of real-valued functions u on I with $u^{(n-1)}$ absolutely continuous on I.

In this paper we establish the existence of extremal solutions to (1.1) when the map F is isotone increasing in x. Our technique involves combining the method of upper and lower solutions [6] with a lattice fixed point theorem [4]. The results differ from the usual theory in the literature [1, 2, 5] since we do not require any type of continuity on F and we do not need to assume that the values of F are convex.

For the remainder of this section we present some definitions and some known facts. Let $X = C(I)$ denote the space of continuous real-valued functions on I with norm $|x|_\infty = \sup_{t \in I} |x(t)|$. We introduce an order relation $\leq$ in X as follows: if $x, y \in X$, then $x \leq y$ means $x(t) \leq y(t)$ for $t \in I$. It is well known that $(X, \leq)$ is a complete lattice [3]. For $x \in X$, by $x \leq A$ we mean $x \leq a$ for every $a \in A$, and by $A \leq B$ we mean $a \leq b$ for every $a \in A$ and $b \in B$. A multivalued map $T : X \to 2^X$

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(here 2^X denotes the class of nonempty subsets of X) is said to be isotone increasing if for $x, y \in X$, $x \leq y$ and $x \neq y$, we have $Tx \leq Ty$.

We now state the lattice fixed point result [4] which will be needed in Section 2 (for completeness we supply the proof).

Theorem 1.1. *Let $(L, \leq)$ be a complete lattice and $T : L \to 2^L$ a multivalued map. Suppose the following conditions hold:*

(i) *T is isotone increasing on L; and*
(ii) $\inf Tx \in Tx$, $\sup Tx \in Tx$ *for each $x \in L$.*

Then the set $P = \{u \in L : u \in Tu\}$ is nonempty and there exists $u_1, u_2 \in L$ with $u_1 \in Tu_1$, $u_2 \in Tu_2$ with

$$u_1 \leq u \leq u_2 \ \ \textit{for all} \ \ u \in L \ \ \textit{with} \ \ u \in Tu.$$

Proof. Define a single valued map $f : L \to L$ by $f(x) = \sup Tx$. Clearly f is isotone increasing on L, so Tarski's fixed point theorem guarantees that

$$P_0 = \{x \in L : \ x = f(x)\} \ \text{ is nonempty and } \ (P_0, \leq) \ \text{ is a complete lattice.}$$

Thus there exists $u_2 \in P_0$ with

$$w \leq u_2 \ \text{ for all } \ w \in L \ \text{ with } \ w = f(w). \tag{1.2}$$

In addition since

$$u_2 = f(u_2) = \sup\, T\,u_2 \ \ \text{ and } \ \ \sup\, T\,u_2 \in T\,u_2,$$

we have $u_2 \in Tu_2$.

Let $q = \sup L$. Take any $u \in L$ with $u \in Tu$ and let us look at the lattice interval $[u, q]$ (which is of course a complete lattice). Clearly $f : [u, q] \to [u, q]$ (since $u \in Tu$ and $f(x) = \sup Tx$), so Tarski's fixed point theorem guarantees that there exists $y_u \in [u, q]$ with $y_u = f(y_u)$. This together with (1.2) yields

$$u \leq y_u \leq u_2.$$

We can do this for all $u \in L$ with $u \in Tu$. We have thus shown that there exists $u_2 \in L$ with $u_2 \in Tu_2$ and

$$u \leq u_2 \ \text{ for all } \ u \in L \ \text{ with } \ u \in T\,u.$$

Similarly we can show that there exists $u_1 \in L$ with $u_1 \in Tu_1$ and

$$u_1 \leq u \ \text{ for all } \ u \in L \ \text{ with } \ u \in T\,u.$$

Hence the proof is finished. □

2. EXISTENCE OF EXTREMAL SOLUTIONS

In this section we establish the existence of extremal solutions to (1.1). Throughout this section we let

$$|F(t,x)| = \{|u| : \ u \in F(t,x)\} \ \text{ for } \ (t,x) \in I \times \mathbb{R},$$

Also we will assume that the following condition is satisfied:

(H1) For any $r > 0$ there exists $h_r \in L^1(I)$ such that $|x| \le r$ implies $|F(t,x)| \le h_r(t)$ for a.e. $t \in I$.

Let the Niemytzky operator be defined by

$$S_F\,(x) = \{v \in L^1(I) : \ v(s) \in F(s, x(s)) \ \text{ for a.e. } \ s \in I\}$$

for $x \in C(I)$.

We now give the definition of upper and lower solutions.

Definition 2.1. A function $\alpha \in AC^n(I)$ is called a lower solution of (1.1) if for any $v \in S_F(\alpha)$ we have $\alpha^{(n)}(t) \le v(t)$ for a.e. $t \in I$ and $\alpha^{(i)}(0) \le x_i$ for $i \in \{0,1,\dots,n-1\}$. A function $\beta \in AC^n(I)$ is called a upper solution of (1.1) if for any $v \in S_F(\beta)$ we have $v(t) \le \beta^{(n)}(t)$ for a.e. $t \in I$ and $x_i \le \beta^{(i)}(0)$ for $i \in \{0,1,\dots,n-1\}$.

Definition 2.2. A real-valued continuous function $\sigma \in L$ (L will be be specified later) on I is said to be a maximal solution of (1.1) in L if it satisfies (1.1) on I and for any other solution $x \in L$ of (1.1) on I we have $x(t) \le \sigma(t)$ for $t \in I$. Similarly we can define a minimal solution $\rho \in L$ of (1.1) in L.

Suppose the following conditions are satisfied:

(H2) $F(t,x)$ is closed for each $(t,x) \in I \times \mathbb{R}$;
(H3) $F(t,x)$ is isotone increasing in x for a.e. $t \in I$ i.e., for $x, y \in \mathbb{R}$ with $x < y$ we have $F(t,x) \le F(t,y)$ for a.e. $t \in I$;
(H4) $S_F(x) \ne \emptyset$ for any $x \in C(I)$; and
(H5) there exists a lower solution α and a upper solution β to (1.1) with $\alpha \le \beta$ on I.

Remark 2.3. Condition (H4) has been discussed extensively in the literature; see [1] and the references therein.

Let

$$L = \{x \in X : \ \alpha(t) \le x(t) \le \beta(t) \ \text{ for } \ t \in I\}.$$

Theorem 2.4. *Let* $F : I \times \mathbb{R} \to 2^{\mathbb{R}}$ *and assume* (H1)–(H5) *hold. Then* (1.1) *has maximal and minimal solutions in* L.

Proof. Now $L = [\alpha, \beta]$ is a lattice interval of the complete lattice X, so $(L, \leq)$ is a complete lattice. Define a multifunction $T : L \to 2^X$ by (here $x \in L$)

$$\begin{aligned} T\,x &= \left\{ u \in X : \; u(t) = \sum_{i=0}^{n-1} \frac{x_i\, t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\, v(s)\, ds, \;\; v \in S_F(x) \right\} \\ &= K \circ S_F(x), \end{aligned}$$

where the operator $L^1(I) \to C(I)$ is defined by

$$K\, y(t) = \sum_{i=0}^{n-1} \frac{x_i\, t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\, y(s)\, ds.$$

We wish to apply Theorem 1.1. First we show T is isotone increasing on L. Let $x, y \in L$ be such that $x \leq y$ and $x \neq y$. Let $u_1 \in Tx$ and $u_2 \in Ty$. Then there exists $v_1 \in S_F(x)$ and $v_2 \in S_F(y)$ with $u_1(t) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v_1(s) ds$ and $u_2(t) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v_2(s) ds$. Now (H3) implies that

$$F(s, x(s)) \leq F(s, y(s)) \quad \text{for a.e.} \quad s \in I,$$

so for $t \in I$ we have

$$\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\, v_1(s)\, ds \leq \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\, v_2(s)\, ds.$$

As a result

$$\begin{aligned} u_1(t) &= \sum_{i=0}^{n-1} \frac{x_i\, t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\, v_1(s)\, ds \\ &\leq \sum_{i=0}^{n-1} \frac{x_i\, t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\, v_2(s)\, ds = u_2(t) \end{aligned}$$

for $t \in I$. Hence $Tx \leq Ty$, i.e., T is isotone increasing on L.

Next we show $T : L \to 2^L$. To see this let $x \in L$ (so in particular $\alpha(t) \leq x(t) \leq \beta(t)$ for $t \in I$). For each $u \in T\beta$ there exists $v \in S_F(\beta)$ (i.e., $v \in L^1(I)$ with $v(s) \in F(s, \beta(s))$ for a.e. $s \in I$) with $u(t) = \sum_{i=0}^{n-1} \frac{x_i t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v(s) ds$. Now since β is an upper solution for (1.1) we have $v(s) \leq \beta^{(n)}(s)$ for a.e. $s \in I$. As a result for $t \in I$ we have

$$\begin{aligned} u(t) &= \sum_{i=0}^{n-1} \frac{x_i\, t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\, v(s)\, ds \\ &\leq \sum_{i=0}^{n-1} \frac{x_i\, t^i}{i!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\, \beta^{(n)}(s)\, ds \\ &= \sum_{i=0}^{n-1} \frac{x_i\, t^i}{i!} + \beta(t) - \sum_{i=0}^{n-1} \frac{\beta^{(i)}(0)\, t^i}{i!} \leq \beta(t). \end{aligned}$$

Consequently $T\beta \le \beta$. A similar argument guarantees that $\alpha \le T\alpha$. Now since T is isotone increasing on L and $\alpha \le x \le \beta$ we have $\alpha \le T\alpha \le Tx \le T\beta \le \beta$, so $T : L \to 2^L$.

Finally we notice that (H2) implies for $x \in L$ that Tx is a closed subset of L. This is clear if we show the values of the Niemytzky operator are closed in $L^1(I)$ (and to see this note if $w_n \to w$ in $L^1(I)$ then $w_n \to w$ in measure, so there exists a subsequence S of integers with w_n converging a.e. to w as $n \to \infty$ through S; now it is clear since (H2) holds that the values of S_F are closed in $L^1(I)$). Thus for each $x \in L$ we have that Tx is a nonempty, closed and bounded subset of L, so as a result we have that $\sup Tx \in Tx$ (also $\inf Tx \in Tx$).

Theorem 1.1 guarantees that the fixed point set of T is nonempty and that it has maximal and minimal elements. This implies that (1.1) has a maximal and minimal solution in L. □

Next we discuss a special case of (1.1), namely

$$\begin{cases} x' \in F(t,x) \quad \text{a.e.} \quad t \in I \\ x(0) = x_0 \in \mathbb{R}. \end{cases} \tag{2.1}$$

In our next result we suppose (H2), (H3) and (H4) hold and in addition we assume the following condition is satisfied:

(H6) $|F(t,y)| \le q(t)\psi(|y|)$ for a.e. $t \in I$ and $y \in \mathbb{R}$ with $\psi : [0,\infty) \to (0,\infty)$ continuous, $q \in L^1(I)$, ψ nondecreasing and $\int_0^a q(t)dt < \int_{|x_0|}^\infty \frac{du}{\psi(u)}$.

Let

$$L = \{x \in X : \; \alpha(t) \le x(t) \le \beta(t) \;\; \text{for} \;\; t \in I\},$$

where

$$\alpha(t) = -\, J^{-1}\left(\int_0^t q(x)\,dx\right) \quad \text{and} \quad \beta(t) = J^{-1}\left(\int_0^t q(x)\,dx\right)$$

with

$$J(z) = \int_{|x_0|}^z \frac{du}{\psi(u)}.$$

Theorem 2.5. *Let $F : I \times \mathbb{R} \to 2^{\mathbb{R}}$ and assume* (H2), (H3), (H4) *and* (H6) *hold. Then* (2.1) *has maximal and minimal solutions in L.*

Proof. The result follows from Theorem 2.4 once we show α is a lower solution of (2.1) and β is an upper solution of (2.1). Notice for a.e. $t \in I$ that

$$\beta'(t) = q(t)\,\psi(\beta(t)) \quad \text{and} \quad -\,\alpha'(t) = q(t)\,\psi(-\,\alpha(t)).$$

Also notice $\beta(0) = |x_0| \ge x_0$ and $\alpha(0) = -|x_0| \le x_0$. Now let $v \in S_F(\beta)$. Then $v \in L^1(I)$ and $v(s) \in F(s, \beta(s))$ for a.e. $s \in I$. Also (H6) implies

$$|v(s)| \le q(s)\,\psi(|\beta(s)|) = q(s)\,\psi(\beta(s)) = \beta'(s) \quad \text{for a.e.} \quad s \in I,$$

so as a result

$$v(s) \le |v(s)| \le \beta'(s) \quad \text{for a.e.} \quad s \in I.$$

Thus β is a upper solution of (2.1). Next let $v \in S_F(\alpha)$. Now (H6) implies

$$-v(s) \le |v(s)| \le q(s)\,\psi(|\alpha(s)|) = q(s)\,\psi(-\alpha(s)) = -\alpha'(s) \quad \text{for a.e.} \quad s \in I,$$

so as a result

$$\alpha'(s) \le v(s) \quad \text{for a.e.} \quad s \in I.$$

Thus α is a upper solution of (2.1). Apply Theorem 2.4. □

In fact the assumptions in Theorem 2.5 guarantee a little more. Before we discuss this we first present a result which will be needed in the proof.

Theorem 2.6. *Let* $F : I \times \mathbb{R} \to 2^{\mathbb{R}}$ *and assume* (H6) *holds. Then any solution of* (2.1) *is in* L *(as described before Theorem 2.5).*

Proof. Let x be any solution of (2.1). We must show

$$|x(t)| \le J^{-1}\left(\int_0^t q(x)\,dx\right) \quad \text{for} \quad t \in I. \tag{2.2}$$

Fix $t \in I$. If $|x(t)| \le |x_0|$, then clearly (2.2) is satisfied. So it remains to consider the case when $|x(t)| > |x_0|$. If $|x(t)| > |x_0|$ then, in view of the initial data, there exists an interval $[b, t] \subseteq I$ with

$$|x(s)| > |x_0| \quad \text{for} \quad s \in (b, t] \quad \text{and} \quad |x(b)| = |x_0|.$$

If $s \in (b, t]$ then $x(s) \neq 0$ and so $|x(s)|' \le |x'(s)|$. This together with (H6) yields

$$|x(s)|' \le q(s)\,\psi(|x(s)|) \quad \text{for} \quad s \in (b, t].$$

Divide by $\psi(|x(s)|)$ and integrate from b to t to obtain

$$J(|x(t)|) \le \int_b^t q(s)\,ds \le \int_0^t q(s)\,ds.$$

That is

$$|x(t)| \le J^{-1}\left(\int_0^t q(s)\,ds\right),$$

so (2.2) is true in this case also. □

Definition 2.7. A real-valued continuous function σ on I is said to be a maximal solution of (2.1) if it satisfies (2.1) on I and for any other solution x of (2.1) on I we have $x(t) \le \sigma(t)$ for $t \in I$. Similarly we can define a minimal solution ρ of (2.1).

Theorem 2.8. *Let* $F : I \times \mathbb{R} \to 2^{\mathbb{R}}$ *and assume* (H2), (H3), (H4) *and* (H6) *hold. Then* (2.1) *has maximal and minimal solutions.*

Proof. From Theorem 2.5 we know that there exist solutions $\rho \in L$ (as described before Theorem 2.5), $\sigma \in L$ of (2.1) with

$$\rho(t) \leq y(t) \leq \sigma(t), \ t \in I \ \text{ for all solution } \ y \in L \ \text{ of } \ (2.1). \tag{2.3}$$

Let x be any solution of (2.1). Theorem 2.6 guarantees that $x \in L$. Now (2.3) implies $\rho(t) \leq x(t) \leq \sigma(t)$ for $t \in I$. □

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