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GREEN'S FUNCTION FOR AN EVEN ORDER MIXED DERIVATIVE PROBLEM ON TIME SCALES

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ABSTRACT. Green's function for an even-order focal problem, where the derivatives alternate between nabla and delta derivatives, is found, and several examples are given for standard time scales. The signs of the function and its derivatives are determined, and whether a symmetry condition holds for an arbitrary time scale is also discussed. The results are then applied to give existence criteria for a positive solution to a nonlinear boundary-value problem.

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1. PRELIMINARIES

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the reals $\mathbb{R}$ [5, 6]. We define the sets $\mathbb{T}^\kappa$ and $\mathbb{T}_\kappa$ by

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} - \{m_1\} & \text{if } \mathbb{T} \text{ has a left scattered maximum } m_1 \\ \mathbb{T} & \text{otherwise,} \end{cases}$$

$$\mathbb{T}_\kappa = \begin{cases} \mathbb{T} - \{m_2\} & \text{if } \mathbb{T} \text{ has a right scattered minimum } m_2 \\ \mathbb{T} & \text{otherwise.} \end{cases}$$

In addition we use the notation $\mathbb{T}^{\kappa^2} = (\mathbb{T}^\kappa)^\kappa$, etc.

Definition 1.1. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

It is convenient to have the graininess operators $\mu_\sigma, \mu_\rho : \mathbb{T} \to [0, \infty)$ defined by $\mu_\sigma(t) = \sigma(t) - t$ and $\mu_\rho(t) = \rho(t) - t$.

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Definition 1.2 (Delta Derivative)**.** Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \le \epsilon|\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

We call $f^\Delta(t)$ the delta derivative of f at t.

Definition 1.3 (Nabla Derivative)**.** For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$, we define $f^\nabla(t)$, to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$|f(\rho(t)) - f(s) - f^\nabla(t)[\rho(t) - s]| \le \epsilon|\rho(t) - s| \quad \text{for all} \quad s \in U.$$

We call $f^\nabla(t)$ the nabla derivative of f at t.

In the case $\mathbb{T} = \mathbb{R}$, $f^\Delta(t) = f'(t) = f^\nabla(t)$. When $\mathbb{T} = \mathbb{Z}$, $f^\Delta(t) = f(t+1) - f(t)$ and $f^\nabla(t) = f(t) - f(t-1)$. One can also define integration on an appropriate class of functions.

Definition 1.4 (Delta Integral)**.** Let $f : \mathbb{T} \to \mathbb{R}$ be a function, and $a, b \in \mathbb{T}$. If there exists a function $F : \mathbb{T} \to \mathbb{R}$ such that $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}$, then F is said to be a delta antiderivative of f. In this case the integral is given by the formula

$$\int_a^b f(\tau)\Delta\tau = F(b) - F(a) \quad \text{for} \quad a, b \in \mathbb{T}.$$

Definition 1.5 (Nabla Integral)**.** Let $f : \mathbb{T} \to \mathbb{R}$ be a function, and $a, b \in \mathbb{T}$. If there exists a function $F : \mathbb{T} \to \mathbb{R}$ such that $F^\nabla(t) = f(t)$ for all $t \in \mathbb{T}$, then F is said to be a nabla antiderivative of f. In this case the integral is given by the formula

$$\int_a^b f(\tau)\nabla\tau = F(b) - F(a) \quad \text{for} \quad a, b \in \mathbb{T}.$$

Definition 1.6. A function $f : \mathbb{T} \to \mathbb{R}$ is called right dense continuous (rd-continuous) provided it is continuous at all right dense points of $\mathbb{T}$ and its left sided limit exists (finite) at left dense points of $\mathbb{T}$. The set of all rd-continuous functions on $\mathbb{T}$ is denoted by

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

Similarly, a function $f : \mathbb{T} \to \mathbb{R}$ is called left dense continuous (ld-continuous) provided it is continuous at all left dense points of $\mathbb{T}$, and its right sided limit exists (finite) at right dense points of $\mathbb{T}$. The set of all ld-continuous functions is denoted

$$C_{ld} = C_{ld}(\mathbb{T}) = C_{ld}(\mathbb{T}, \mathbb{R}).$$

Remark 1.7. All rd-continuous functions are delta integrable, and all ld-continuous functions are nabla integrable.

2. GREEN'S FUNCTION

In order to simplify the notation, let $x^\star := (x^\nabla)^\Delta$, $(x^{\star^2}) := (((x^\nabla)^\Delta)^\nabla)^\Delta$, etc. We consider the $2n^{th}$ order boundary value problem

$$L_{2n}x := (-1)^n x^{\star^n} = h$$

$$x^{\star^i}(a) = 0 \quad \text{for } 0 \le i \le n-1 \tag{2.1}$$

$$(x^{\star^i})^\nabla(b) = 0 \quad \text{for } 0 \le i \le n-1 \tag{2.2}$$

on a time scale $\mathbb{T}$, where $a \in \mathbb{T}_{\kappa^n}$, $b \in \mathbb{T}^{\kappa^n}$, $\sigma^n(a) < \rho^n(b)$, $x : [\rho^n(a), \sigma^n(b)] \to \mathbb{R}$, and $h : [a, b] \to \mathbb{R}$ is a given rd-continuous function. A form of this problem was first discussed with $n = 1$ in [4, p.91]. For more on Green's functions for higher-order boundary-value problems on time scales, see [2, 7, 8].

The proofs of the following two lemmas are standard.

Lemma 2.1. *The homogeneous boundary value problem $L_{2n}x = 0$, (2.1), (2.2) has only the trivial solution.*

Lemma 2.2. *The nonhomogeneous boundary value problem*

$$L_{2n}x = h$$

$$x^{\star^i}(a) = \alpha_i \quad \textit{for } 0 \le i \le n-1$$

$$(x^{\star^i})^\nabla(b) = \beta_i \quad \textit{for } 0 \le i \le n-1,$$

where $\alpha_i, \beta_i \in \mathbb{R}$ for $0 \le i \le n-1$ and h is a given rd-continuous function, has a unique solution.

We define the Cauchy function for this boundary value problem as follows.

Definition 2.3. The function $y : [\rho^n(a), \sigma^n(b)] \times [a, b] \to \mathbb{R}$ is the Cauchy function for $L_{2n}x = 0$ provided for each fixed $s \in [a, b]$, $y(\cdot, s)$ is the solution to the initial value problem

$$L_{2n}y(\cdot, s) = 0$$

$$y^{\star^i}(s, s) = 0 \quad \text{for } 0 \le i \le n-1 \tag{2.3}$$

$$(y^{\star^i})^\nabla(s, s) = 0 \quad \text{for } 0 \le i \le n-2 \tag{2.4}$$

$$(y^{\star^{n-1}})^\nabla(s, s) = (-1)^n. \tag{2.5}$$

Remark 2.4. In order to make calculations with the Cauchy and Green's functions simpler, we define the functions $d_i : [\rho^n(a), \sigma^n(b)] \times [a, b] \to \mathbb{R}$ for $i \geq 0$ recursively by

$$\begin{aligned} d_0(t,s) &\equiv 1 \\ d_{2i+1}(t,s) &= \int_s^t d_{2i}(\tau,s)\,\nabla\tau \\ d_{2i}(t,s) &= \int_s^t d_{2i-1}(\tau,s)\,\Delta\tau. \end{aligned}$$

If $i < 0$, then $d_i(t,s)$ is taken to be identically zero. The idea for these functions comes from similarly defined functions for the delta case [1, 5] and the nabla case [3, 6].

Example 2.5. For $\mathbb{T} = \mathbb{R}$,

$$d_i(t,s) = \frac{(t-s)^i}{i!}.$$

However if $\mathbb{T} = \mathbb{Z}$,

$$d_{2i}(t,s) = \frac{(t-s+i-1)^{(2i)}}{(2i)!}$$

and

$$d_{2i+1}(t,s) = \frac{(t-s+i)^{(2i+1)}}{(2i+1)!}.$$

Example 2.6. Consider the time scale $\mathbb{T} = \{q^n : n \in \mathbb{N}\}$ where $q > 1$. Here we get

$$\begin{aligned} d_0(t,s) &= 1 \\ d_1(t,s) &= t-s \\ d_2(t,s) &= \frac{(t-s)(t-qs)}{1+q} \\ d_3(t,s) &= \frac{(t-s)(t-qs)(tq-s)q}{(1+q)(1+q+q^2)}. \end{aligned}$$

In fact,

$$d_{2n}(t,s) = d_{2n-1}(t,s)\frac{t-q^n s}{\sum_{i=0}^{2n-1} q^i}$$

and

$$d_{2n+1}(t,s) = d_{2n}(t,s)\frac{(tq^n-s)q^n}{\sum_{i=0}^{2n} q^i}.$$

Remark 2.7. These recursively defined functions have several useful properties which can be proven using standard theorems for time scales [5, 6]. Here we take $i \geq 1$, except for properties 2 and 4 where $i \geq 0$.

1. $d_{2i}^{\Delta}(t,s) = d_{2i-1}(t,s)$
2. $d_{2i+1}^{\nabla}(t,s) = d_{2i}(t,s)$
3. $d_{2i}(\sigma(t),s) = d_{2i}(t,s) + \mu_\sigma(t)d_{2i-1}(t,s)$

4. $d_{2i+1}(\rho(t), s) = d_{2i+1}(t, s) + \mu_\rho(t) d_{2i}(t, s)$
5. $d_{2i}(\rho(t), s) = d_{2i}(t, s) + \mu_\rho(t)\left[d_{2i-1}(t, s) + \mu_\rho(t) d_{2i-2}(t, s)\right]$
6. $d_{2i+1}(\sigma(t), s) = d_{2i+1}(t, s) + \mu_\sigma(t)\left[d_{2i}(t, s) + \mu_\sigma(t) d_{2i-1}(t, s)\right]$.

Lemma 2.8. *The Cauchy function for $L_{2n}x = 0$ is $(-1)^n d_{2n-1}(t, s)$.*

Proof. By definition $d_{2n-1}(\cdot, s)$ satisfies $L_{2n} d_{2n-1}(\cdot, s) = 0$ for any fixed $s \in [a, b]$. Thus it only remains to show that $d_{2n-1}(\cdot, s)$ satisfies the initial values (2.3), (2.4), and (2.5). For $0 \le i \le n-1$,

$$(-1)^n d_{2n-1}^{\star^i}(t, s) = (-1)^n d_{2n-2i-1}(t, s),$$

so $(-1)^n d_{2n-1}(t, s)$ satisfies the initial conditions (2.3). In addition for $0 \le i \le n-2$

$$((-1)^n d_{2n-1}^{\star^i})^\nabla(t, s) = (-1)^n d_{2n-2i-2}(t, s),$$

so the initial conditions (2.4) are satisfied. For condition (2.5), note that

$$((-1)^n d_{2n-1}^{\star^{n-1}})^\nabla(t, s) = (-1)^n d_0(t, s) = (-1)^n,$$

which completes the proof. □

Theorem 2.9. *For each fixed $s \in [a, b]$ let $u(\cdot, s)$ be the unique solution of the boundary value problem*

$$L_{2n} u(\cdot, s) = 0$$

$$u^{\star^i}(a, s) = 0 \quad \text{for } 0 \le i \le n-1 \tag{2.6}$$

$$(u^{\star^i})^\nabla(b, s) = -(-1)^n d_{2n-2i-2}(b, s) \quad \text{for } 0 \le i \le n-1. \tag{2.7}$$

Then the Green's function $G : [\rho^n(a), \sigma^n(b)] \times [a, b] \to \mathbb{R}$ for $L_{2n}x = 0$, (2.1), (2.2) is given by

$$G(t, s) = \begin{cases} u(t, s) & : t \le s \\ u(t, s) + (-1)^n d_{2n-1}(t, s) & : s \le t. \end{cases}$$

Proof. Note that since for each fixed $s \in [a, b]$, $u(\cdot, s)$ and $(-1)^n d_{2n-1}(\cdot, s)$ are solutions of $L_{2n}x = 0$, and

$$v(t, s) := u(t, s) + (-1)^n d_{2n-1}(t, s)$$

is also a solution for each fixed $s \in [a, b]$. Define x by

$$x(t) := \int_a^b G(t, s) h(s)\, \Delta s,$$

where $h : [a, b] \to \mathbb{R}$ is a given rd-continuous function. One may rewrite $x(t)$ in the following manner.

$$\begin{aligned}
x(t) &= \int_a^b G(t,s)h(s)\,\Delta s \\
&= \int_a^t G(t,s)h(s)\,\Delta s + \int_t^b G(t,s)h(s)\,\Delta s \\
&= \int_a^t (u(t,s) + (-1)^n d_{2n-1}(t,s))h(s)\,\Delta s + \int_t^b u(t,s)h(s)\,\Delta s \\
&= \int_a^b u(t,s)h(s)\,\Delta s + \int_a^t (-1)^n d_{2n-1}(t,s)h(s)\,\Delta s.
\end{aligned}$$

Note that $x(a) = 0$ using boundary condition (2.6). Now taking the nabla derivative of x and using property 2 of Remark 2.7, we find

$$\begin{aligned}
x^\nabla(t) &= \left[\int_a^b u(t,s)h(s)\,\Delta s + \int_a^t (-1)^n d_{2n-1}(t,s)h(s)\,\Delta s\right]^\nabla \\
&= \int_a^b u^\nabla(t,s)h(s)\,\Delta s + \int_a^t (-1)^n d_{2n-1}^\nabla(t,s)h(s)\,\Delta s \\
&\quad +(-1)^n d_{2n-1}(\rho(t),\rho(t))h(\rho(t)) \\
&= \int_a^b u^\nabla(t,s)h(s)\,\Delta s + \int_a^t (-1)^n d_{2n-2}(t,s)h(s)\,\Delta s,
\end{aligned}$$

which indicates that $x^\nabla(b) = 0$ by boundary condition (2.7). Thus for $1 \le i \le n-1$,

$$\begin{aligned}
(x^{\star^i})^\nabla(t) &= \left[\int_a^b u^{\star^i}(t,s)h(s)\,\Delta s + \int_a^t (-1)^n d_{2n-2i-1}(t,s)h(s)\,\Delta s\right]^\nabla \\
&= \int_a^b (u^{\star^i})^\nabla(t,s)h(s)\,\Delta s + \int_a^t (-1)^n d_{2n-2i-1}^\nabla(t,s)h(s)\,\Delta s \\
&\quad +(-1)^n d_{2n-2i-1}(\rho(t),\rho(t))h(\rho(t)) \\
&= \int_a^b (u^{\star^i})^\nabla(t,s)h(s)\,\Delta s + \int_a^t (-1)^n d_{2n-2i-2}(t,s)h(s)\,\Delta s,
\end{aligned}$$

whence x satisfies boundary conditions (2.2). On the other hand, for $1 \le i \le n-1$,

$$\begin{aligned}
x^{\star^i}(t) &= \left[\int_a^b (u^{\star^{i-1}})^{\nabla}(t,s)h(s)\,\Delta s + \int_a^t (-1)^n d_{2n-2-2(i-1)}(t,s)h(s)\,\Delta s\right]^{\Delta} \\
&= \int_a^b u^{\star^i}(t,s)h(s)\,\Delta s + \int_a^t (-1)^n d^{\Delta}_{2n-2i}(t,s)h(s)\,\Delta s \\
&\quad +(-1)^n d_{2n-2i}(\sigma(t),t)h(t) \\
&= \int_a^b u^{\star^i}(t,s)h(s)\,\Delta s + \int_a^t (-1)^n d_{2n-2i-1}(t,s)h(s)\,\Delta s \\
&\quad +(-1)^n h(t)[d_{2n-2i}(t,t) + \mu_\sigma(t) d_{2n-2i-1}(t,t)] \\
&= \int_a^b u^{\star^i}(t,s)h(s)\,\Delta s + \int_a^t (-1)^n d_{2n-2i-1}(t,s)h(s)\,\Delta s,
\end{aligned}$$

so x satisfies boundary conditions (2.1). To see that x is a solution of $L_{2n}x = h$, note that by properties of the Cauchy function and $u(\cdot,s)$ we have

$$\begin{aligned}
(-1)^n x^{\star^n}(t) &= (-1)^n \int_a^b u^{\star^n}(t,s)h(s)\,\Delta s + (-1)^n \left[\int_a^t (-1)^n d_0(t,s)h(s)\,\Delta\right]^{\Delta} \\
&= \int_a^b (L_{2n}u(t,s))h(s)\,\Delta s + (-1)^n(-1)^n h(t) \\
&= h(t).
\end{aligned}$$

□

Theorem 2.10. *Let*

$$u(t,s) = \begin{vmatrix}
0 & d_1(t,a) & d_3(t,a) & \dots & d_{2n-3}(t,a) & d_{2n-1}(t,a) \\
(-1)^n d_{2n-2}(b,s) & 1 & d_2(b,a) & \dots & d_{2n-4}(b,a) & d_{2n-2}(b,a) \\
(-1)^n d_{2n-4}(b,s) & 0 & 1 & \dots & d_{2n-6}(b,a) & d_{2n-4}(b,a) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(-1)^n d_2(b,s) & 0 & 0 & \dots & 1 & d_2(b,a) \\
(-1)^n & 0 & 0 & \dots & 0 & 1
\end{vmatrix}.$$

Then the Green's function for the boundary value problem $L_{2n}x = 0$, (2.1), (2.2) is given by

$$G(t,s) = \begin{cases} u(t,s) & : t \le s \\ u(t,s) + (-1)^n d_{2n-1}(t,s) & : s \le t. \end{cases}$$

Proof. By Theorem 2.9 it suffices to show that for any fixed $s \in [a,b]$, $u(\cdot,s)$ satisfies the boundary value problem $L_{2n}u(\cdot,s) = 0$, (2.6), (2.7). Clearly $u(\cdot,s)$ satisfies the boundary conditions (2.6) for any fixed $s \in [a,b]$ by definition. In addition $u(\cdot,s)$ is a solution of $L_{2n}u(\cdot,s) = 0$ for any fixed $s \in [a,b]$. Thus it remains to show that the boundary conditions (2.7) are satisfied. Let

$$v(t,s) := u(t,s) + (-1)^n d_{2n-1}(t,s).$$

Then boundary conditions (2.7) are equivalent to showing that $v(\cdot,s)$ satisfies the boundary conditions (2.2) for any fixed $s\in[a,b]$. However, the first and $(i+2)^{nd}$ rows of the matrix $(v^{\star^i})^{\nabla}(b,s)$ are the same for any fixed $s\in[a,b]$, $0\le i\le n-1$. Thus $v(\cdot,s)$ satisfies the boundary conditions (2.2) for any fixed $s\in[a,b]$. □

Lemma 2.11. *The Green's function satisfies*

$$G(t,s)=G(s,t)$$

if and only if the Cauchy function satisfies

$$d_{2n-1}(t,s)=-d_{2n-1}(s,t).$$

In particular, $G(t,s)=G(s,t)$ *for* $\mathbb{T}=\mathbb{R}$, $h\mathbb{Z}$, $q^{\mathbb{N}}$.

Proof. Let

$$v(t,s):=u(t,s)+(-1)^n d_{2n-1}(t,s).$$

Then $G(t,s)=G(s,t)$ if and only if $u(t,s)=v(s,t)$ and $u(s,t)=v(t,s)$. But this happens if and only if

$$\begin{aligned} u(t,s) &= u(s,t)+(-1)^n d_{2n-1}(s,t)\\ &= v(t,s)+(-1)^n d_{2n-1}(s,t)\\ &= u(t,s)+(-1)^n d_{2n-1}(t,s)+(-1)^n d_{2n-1}(s,t). \end{aligned}$$

The proof is complete. □

Example 2.12. Suppose $\mathbb{T}=\mathbb{Z}$. For $n=2$,

$$\begin{aligned} u(t,s) &= \begin{vmatrix} 0 & d_1(t,a) & d_3(t,a)\\ d_2(b,s) & 1 & d_2(b,a)\\ 1 & 0 & 1 \end{vmatrix}\\ &= d_1(t,a)d_2(b,a)-d_3(t,a)-d_1(t,a)d_2(b,s)\\ &= \frac{1}{6}(t-a)(-t^2+2at+6bs-3s-3s^2-6ab+2a^2+3a+1), \end{aligned}$$

and

$$v(t,s)=\frac{1}{6}(s-a)(-s^2+2as+6bt-3t-3t^2-6ab+2a^2+3a+1).$$

Notice that $G(t,s)=G(s,t)$.

Example 2.13. Suppose $\mathbb{T}=q^{\mathbb{N}_0}$. For $n=2$,

$$\begin{aligned} u(t,s) &= \frac{(t-a)}{(1+q)(1+q+q^2)}(-2bqa-2bq^2a-bq^3a-ab+qa^2+q^3a^2-t^2q^2\\ &\quad +qta+q^3at+2bqs+2bq^2s+bq^3s+bs-qs^2-q^2s^2-q^3s^2) \end{aligned}$$

and

$$\begin{aligned} v(t,s) &= \frac{(s-a)}{(1+q)(1+q+q^2)}(-2bqa-2bq^2a-bq^3a-ab+qa^2+q^3a^2-s^2q^2 \\ &\quad +qsa+q^3as+2bqt+2bq^2t+bq^3t+bt-qt^2-q^2t^2-q^3t^2). \end{aligned}$$

Again $G(t,s)=G(s,t)$.

Example 2.14. Suppose $\mathbb{T}=[0,1]\cup[2,3]$. Then $d_0(t,s)\equiv 1$ and $d_1(t,s)=t-s$. Suppose $s\in[0,1]$ and $t\in[2,3]$. Then

$$\begin{aligned} d_2(t,s) &= \int_s^t (\tau-s)\Delta\tau \\ &= \left(\int_s^1+\int_1^2+\int_2^t\right)(\tau-s)\Delta\tau \\ &= \frac{1}{2}(1-s)^2+1-s+\frac{1}{2}(t-s)^2-\frac{1}{2}(2-s)^2 \end{aligned}$$

and

$$\begin{aligned} d_3(t,s) &= \int_s^t d_2(\tau,s)\nabla\tau \\ &= \left(\int_s^1+\int_1^2+\int_2^t\right)d_2(\tau,s)\nabla\tau \\ &= \frac{1}{2}(1-s)^3-\frac{1}{2}(2-s)^2(1-s)+(1-s)^2+\frac{1}{6}(1-s)^3+\frac{1}{2}(1-s)^2 \\ &\quad +1-s+\frac{1}{2}(1-s)^2(t-2)-\frac{1}{2}(2-s)^2(t-2)+(1-s)(t-2) \\ &\quad +\frac{1}{6}(t-s)^3-\frac{1}{6}(2-s)^3 \\ &= \frac{1}{6}t^3+\frac{1}{2}ts^2-\frac{1}{2}t^2s-\frac{1}{2}t-\frac{1}{6}s^3+\frac{5}{6}. \end{aligned}$$

A routine calculation shows that

$$d_3(t,s)+d_3(s,t)=\frac{1}{6}(10-3s-3t)\neq 0,$$

so that in general there is no symmetry in the Green's function.

Lemma 2.15. *Let $G(t,s)$ be the Green's function for the boundary value problem given by $L_{2n}x=0$, (2.1), (2.2). Then the following hold:*

$$(-1)^i(G^{\star^i})^\nabla(t,s)>0,\quad a\le t,s<b,\quad 0\le i\le n-2.$$
$$(-1)^iG^{\star^i}(t,s)>0,\quad a<t,s\le b,\quad 0\le i\le n-1.$$

Proof. By the previous theorem, the Green's function for this boundary value problem is given by

$$G(t,s)=\begin{cases} u(t,s) & : t\le s \\ u(t,s)+(-1)^n d_{2n-1}(t,s) & : s\le t, \end{cases}$$

where

$$u(t,s)=\begin{vmatrix} 0 & d_1(t,a) & d_3(t,a) & \dots & d_{2n-3}(t,a) & d_{2n-1}(t,a)\\ (-1)^n d_{2n-2}(b,s) & 1 & d_2(b,a) & \dots & d_{2n-4}(b,a) & d_{2n-2}(b,a)\\ (-1)^n d_{2n-4}(b,s) & 0 & 1 & \dots & d_{2n-6}(b,a) & d_{2n-4}(b,a)\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ (-1)^n d_2(b,s) & 0 & 0 & \dots & 1 & d_2(b,a)\\ (-1)^n & 0 & 0 & \dots & 0 & 1 \end{vmatrix}.$$

Let

$$v(t,s):=u(t,s)+(-1)^n d_{2n-1}(t,s),$$

and fix $s\in[a,b]$. Since $v(\cdot,s)$ satisfies $L_{2n}v(\cdot,s)=0$, $(v^{\star^{n-1}})^\nabla(t,s)$ is a constant. Considering the boundary condition at b, this gives that $(v^{\star^{n-1}})^\nabla(t,s)\equiv 0$, which in turn implies that $(-1)^{n-1}v^{\star^{n-1}}(t,s)$ is a constant. Considering $(u^{\star^{n-1}})^\nabla(t,s)$, we get

$$\begin{aligned}(u^{\star^{n-1}})^\nabla(t,s) &= \begin{vmatrix} 0 & 0 & 0 & \dots & 0 & 1\\ (-1)^n d_{2n-2}(b,s) & 1 & d_2(b,a) & \dots & d_{2n-4}(b,a) & d_{2n-2}(b,a)\\ (-1)^n d_{2n-4}(b,s) & 0 & 1 & \dots & d_{2n-6}(b,a) & d_{2n-4}(b,a)\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ (-1)^n d_2(b,s) & 0 & 0 & \dots & 1 & d_2(b,a)\\ (-1)^n & 0 & 0 & \dots & 0 & 1 \end{vmatrix}\\ &= (-1)^{n+1}\begin{vmatrix} (-1)^n d_{2n-2}(b,s) & 1 & d_2(b,a) & \dots & d_{2n-4}(b,a)\\ (-1)^n d_{2n-4}(b,s) & 0 & 1 & \dots & d_{2n-6}(b,a)\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ (-1)^n d_2(b,s) & 0 & 0 & \dots & 1\\ (-1)^n & 0 & 0 & \dots & 0 \end{vmatrix}\\ &= (-1)^{n+1}.\end{aligned}$$

Thus for $t\le s$,

$$(-1)^{n-1}(G^{\star^{n-1}})^\nabla(t,s)>0.$$

This implies that $(-1)^{n-1}u^{\star^{n-1}}(t,s)$ is increasing. Considering the boundary condition at a, this gives that $(-1)^{n-1}u^{\star^{n-1}}(t,s)>0$ for $a<t\le s$. Since

$$(-1)^{n-1}u^{\star^{n-1}}(s,s)=(-1)^{n-1}v^{\star^{n-1}}(s,s),$$

we have that

$$(-1)^{n-1}v^{\star^{n-1}}(t,s)\equiv\text{constant}>0.$$

This implies that $(-1)^{n-1}(v^{\star^{n-2}})^\nabla(t,s)$ is increasing in t, and is zero at b, thus

$$(-1)^{n-2}(v^{\star^{n-2}})^\nabla(t,s)>0$$

for $s\le t<b$. Again using the fact that

$$(-1)^{n-2}(u^{\star^{n-2}})^\nabla(s,s)=(-1)^{n-2}(v^{\star^{n-2}})^\nabla(s,s),$$

we have

$$(-1)^{n-2}(u^{\star^{n-2}})^{\nabla}(t,s) > 0$$

as well. Continuing in this fashion gives the desired result. □

Remark 2.16. A discussion completely analogous to that given in this paper is possible for the boundary-value problem

$$\begin{aligned}
&(-1)^n x^{(\Delta\nabla)^n} = h \\
&x^{(\Delta\nabla)^i}(a) = 0 \quad \text{ for } 0 \le i \le n-1 \\
&(x^{(\Delta\nabla)^i})^{\Delta}(b) = 0 \quad \text{ for } 0 \le i \le n-1.
\end{aligned}$$

The corresponding Cauchy and Green's functions would be found using the functions

$$c_i : [\rho^n(a), \sigma^n(b)] \times [a,b] \to \mathbb{R}$$

for $i \ge 0$, defined recursively by

$$\begin{aligned}
c_0(t,s) &\equiv 1 \\
c_{2i+1}(t,s) &= \int_s^t c_{2i}(\tau,s)\,\Delta\tau \\
c_{2i}(t,s) &= \int_s^t c_{2i-1}(\tau,s)\,\nabla\tau.
\end{aligned}$$

3. EXISTENCE OF A POSITIVE SOLUTION

We apply the results for the Green's function from the previous section to prove the existence of a positive solution to the nonlinear boundary-value problem $L_{2n}x = f(\cdot, x)$, (2.1), (2.2), where $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is rd-continuous, f nonnegative for $x \ge 0$. The solutions are the fixed points of the operator $\mathcal{A}$ defined by

$$\mathcal{A}x(t) = \int_a^b G(t,s) f(s, x(s))\,\Delta s,$$

where $G(t,s)$ is the Green's function as in Theorem 2.10 for the homogeneous problem $L_{2n}x = 0$, (2.1), (2.2).

We will employ the following fixed point theorem due to Krasnoselskii [9]; first, a few definitions. A nonempty closed convex set $\mathcal{P}$ contained in a real Banach space E is called a cone if it satisfies the following two conditions:

(i) if $x \in \mathcal{P}$ and $\lambda \ge 0$ then $\lambda x \in \mathcal{P}$;
(ii) if $x \in \mathcal{P}$ and $-x \in \mathcal{P}$ then $x = 0$.

The cone $\mathcal{P}$ induces an ordering $\le$ on E by $x \le y$ if and only if $y - x \in \mathcal{P}$. An operator A is said to be completely continuous if it is continuous and compact (maps bounded sets into relatively compact sets).

Theorem 3.1. *Let E be a Banach space, $P \subseteq E$ be a cone, and suppose that Ω_1, Ω_2 are bounded open balls of E centered at the origin, with $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $\mathcal{A} : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator such that either*

$$\|\mathcal{A}u\| \le \|u\|, u \in P \cap \partial\Omega_1 \text{ and } \|\mathcal{A}u\| \ge \|u\|, u \in P \cap \partial\Omega_2$$

or

$$\|\mathcal{A}u\| \ge \|u\|, u \in P \cap \partial\Omega_1 \text{ and } \|\mathcal{A}u\| \le \|u\|, u \in P \cap \partial\Omega_2$$

holds. Then $\mathcal{A}$ has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Let $\mathcal{B}$ denote the Banach space $C_{rd}[\rho^n(a), \sigma^n(b)]$ with the norm

$$\|x\| = \sup_{t\in[a,b]} |x(t)|.$$

By Lemma 2.15, $G(t, s) > 0$ for $a < t, s \le b$ and $G^\nabla(t, s) > 0$ for $a \le t, s < b$. Let $\eta \in (a, b)$. Then

$$0 < G(\eta, s) \le G(t, s) \le G(b, s)$$

for all $s \in (a, b]$, $t \in [\eta, b]$. Set

$$\ell := \min_{s\in[a,b]} \frac{G(\eta, s)}{G(b, s)}. \tag{3.1}$$

Clearly $0 < \ell < 1$ and

$$\ell G(b, s) \le G(t, s) \le G(b, s)$$

for all $t \in [\eta, b]$, $s \in [a, b]$. Define the cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \{x \in \mathcal{B} : x(t) \ge \ell\|x\|, t \in [\eta, b]\},$$

where ℓ is given in (3.1).

In the following discussion we will need the constants

$$m^{-1} := \int_a^b G(b, s)\Delta s \tag{3.2}$$

and

$$r^{-1} := \ell \int_\eta^b G(b, s)\Delta s. \tag{3.3}$$

Then the growth restrictions on f that will yield the existence of a positive solution are as follows:

(C_1) There exists a $p > 0$ such that $f(t, x) \le mp$ for $t \in [a, b]$ and $0 \le x \le p$.
(C_2) There exists a $q > 0$ such that $f(t, x) \ge rx$ for $t \in [\eta, b]$ and $q\ell \le x \le q$.

Theorem 3.2. *Suppose there exist positive numbers $p \ne q$ such that condition (C_1) is satisfied with respect to p and condition (C_2) is satisfied with respect to q. Then $L_{2n}x = f(\cdot, x)$, (2.1), (2.2), has a positive solution x such that $||x||$ lies between p and q.*

Proof. For $x \in \mathcal{P}$, $x(t) \geq \ell\|x\|$ for all $t \in [\eta, b]$. Let $t \in [\eta, b]$. Then

$$\begin{aligned}\mathcal{A}x(t) &= \int_a^b G(t,s)f(s,x(s))\Delta s\\ &\geq \ell \int_a^b G(b,s)f(s,x(s))\Delta s\\ &= \ell||Ax||,\end{aligned}$$

so that $\mathcal{A}(\mathcal{P}) \subset \mathcal{P}$.

Now, without loss of generality, we may assume $0 < p < q$. Define bounded open balls centered at the origin by

$$\Omega_p = \{x \in \mathcal{B} : ||x|| < p\},$$

and

$$\Omega_q = \{x \in \mathcal{B} : ||x|| < q\}.$$

Then $0 \in \Omega_p \subset \Omega_q$. For $x \in \mathcal{P} \cap \partial\Omega_p$ so that $||x|| = p$, we have

$$\begin{aligned}||\mathcal{A}x|| &= \int_a^b G(b,s)f(s,x(s))\Delta s\\ &\leq mp \int_a^b G(b,s)\Delta s\\ &= p\\ &= ||x||\end{aligned}$$

using (C_1) and (3.2). Thus, $||\mathcal{A}x|| \leq ||x||$ for $x \in \mathcal{P} \cap \partial\Omega_p$.

Similarly, let $x \in \mathcal{P} \cap \partial\Omega_q$, so that $||x|| = q$. Then

$$\min_{t\in[\eta,b]} x(t) \geq \ell||x||,$$

so that $q\ell \leq x(s) \leq q$ for $s \in [\eta, b]$, and we have

$$\begin{aligned}||\mathcal{A}x|| &= \int_a^b G(b,s)f(s,x(s))\Delta s\\ &\geq \int_\eta^b G(b,s)f(s,x(s))\Delta s\\ &\geq r \int_\eta^b G(b,s)x(s)\Delta s\\ &\geq q\\ &= ||x||\end{aligned}$$

by (C_2) and (3.3). Consequently, $||\mathcal{A}x|| \geq ||x||$ for $x \in \mathcal{P} \cap \partial\Omega_q$. By Theorem 3.1, $\mathcal{A}$ has a fixed point $x \in \mathcal{P} \cap (\overline{\Omega}_q \setminus \Omega_p)$, which is a positive solution of $L_{2n}x = f(\cdot, x)$, (2.1), (2.2), such that $p \leq ||x|| \leq q$. □

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