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EXPONENTIAL FUNCTIONS ON TIME SCALES: THEIR ASYMPTOTIC BEHAVIOR AND CALCULATION

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ABSTRACT. We consider certain classes of "exponential functions" on time scales, which are defined via dynamic equations and generalize the ordinary exponential function. In this paper we are interested first in the asymptotic behavior of these functions (say compared with the ordinary exponential) and second, how to efficiently calculate matrix analogues of these functions. For the latter question we discuss a recent algorithm introduced by W. A. Harris, Jr. and compare it to the well-known Putzer algorithm.

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1. INTRODUCTION

During the past decade, the use of "time scales" as a means for unifying and extending results about various types of dynamic equations has proven to be both very prolific and fruitful. Many classical results from the theories of differential and difference equations have time scales analogues and this helps to explain from a broad perspective the "raison d'être" for a particular type of method, independent from the particular special case that spawned it. On the other hand, while general aspects may be the same, the structure of each time scale contains characteristic information which can be used to more closely analyze the behavior of functions defined as solutions of dynamic equations. In this paper we will be concerned with such explicit, fine behavior, that concerns a class of special functions and we will investigate how the particular choice of the time scale influences the asymptotic behavior of these functions.

For linear differential equations, one could argue that the exponential is the quintessential function appearing in solutions of the simplest and most natural types and for linear difference equations, the situation is similar. Recently [8, 4, 5] exponential functions for general time scales have been extensively investigated. They are

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nontrivial solutions of the equation

$$y^{\Delta}(t) = \lambda y(t), \tag{1.1}$$

where Δ is the "delta derivative" (see [3, p. 5]) corresponding to the time scale $\mathbb{T}$ and λ is a constant (usually real). For the time scale $\mathbb{R}$, the operator is just the usual derivative d/dt and the solution is $c\,e^{\lambda t}$ while in the case of time scale $\mathbb{N}$, Δ is the forward difference operator and solutions are $c\,e^{t\ln(1+\lambda)}$, where in both cases c is a constant. (Of course for difference equations the constant can be replaced by an arbitrary 1-periodic function, but if we are only interested in the value of the solution on $\mathbb{N}$, this reduces to just a constant multiple.) In most discussions the particular solution of (1.1) is normalized to have the value 1 at some particular point in $\mathbb{T}$, say t_0, and in this case we will use the notation

$$\exp_{\lambda}(t, t_0; \mathbb{T}) \quad \text{ or } \quad e_{\lambda}(t, t_0; \mathbb{T})$$

for the function. Of course, since the time scale is uniquely determined by the forward jump $\sigma(t)$ (see [3, p. 1]) and some initial point t_0 we could just as well use the notation

$$e_{\lambda}(t, t_0; \sigma(t)).$$

One question we are interested in concerns how these functions grow as t tends to infinity. We shall always therefore consider only time scales which are unbounded from above. One motivation for studying the asymptotic behavior comes from a curiosity, but more pragmatic reasons concern applying this information to study the asymptotic behavior of solutions of more complicated equations or systems and their exponential dichotomies and also for applications to problems in control theory.

It is fairly easy to reason that when $\lambda > 0$ and $\sigma(t)$ is large compared to t, then solutions should grow at a slower rate compared to cases when $\sigma(t)$ is of smaller size. One sees this when comparing $e^{\lambda t}$ ($\sigma(t) = t$) with $e^{t\ln(1+\lambda)}$ ($\sigma(t) = t + 1$) and we will see that this trend continues for the class of time scales $\mathbb{N}^p$, where p is a positive real number. For integers $p \geq 2$ the rate of growth will be shown to be sub-exponential. The quantity $\sigma(t) - t$ is called the "graininess" of the time scale and our purpose is to obtain rather exact quantitative information on the asymptotic behavior of solutions and compare those orders of growth with the classical exponential function, as a "function" of the graininess. We will discuss several methods for obtaining the asymptotic representation of various classes of exponential functions on time scales and will identify certain parameters in the graininess which characterize "asymptotic exponential classes" of time scales.

The second question we are concerned with relates to the case when (1.1) represents a system of linear equations and the solutions therefore can be interpreted as matrix exponentials on $\mathbb{T}$. Again in the classical cases the problem of how to calculate the classical matrix exponential has a long history and many procedures and algorithms have been proposed.

Recently Harris, Filmore, and Smith [7] have discussed an algorithm for calculating the classical exponential matrix function through properties of solutions of scalar linear differential equations and which has similarities with the well-studied Putzer algorithm. We show that the so-called Harris algorithm can also be applied to the calculation of matrix exponential functions on general time scales. Finally we remark that the Putzer algorithm can be viewed as a special case of Harris' algorithm.

For a good background for the general theory of dynamic equations on time scales, we refer the reader to the recent book by M. Bohner and A. Peterson with that title [3].

2. ASYMPTOTIC BEHAVIOR OF SCALAR EXPONENTIAL FUNCTIONS

Our goal in this section is to investigate the asymptotic behavior of the scalar function $e_\lambda(t, t_0; \mathbb{T})$ as $t \to \infty$ $(t \in \mathbb{T})$ for a collection of time scales. We assume throughout this section that

$$1 + \mu(t)\lambda \neq 0 \text{ for all } t \in \mathbb{T}$$

and usually that $\lambda \in \mathbb{R}$, although this latter assumption is often not necessary. We recall that, for $t \in \mathbb{T}$, $e_\lambda(t, t_0; \mathbb{T})$ is the unique solution of the IVP

$$y^\Delta = \lambda y \qquad y(t_0) = 1. \tag{2.1}$$

Hilger [8] showed the solution of (2.1) may be represented in the form (if $\mu(t) > 0$ for all $t \in \mathbb{T}$)

$$\exp\left[\int_{t_0}^{t} \frac{1}{\mu(\tau)} \mathrm{Log}(1 + \lambda\mu(t))\, \Delta\tau\right],$$

where $\exp[\cdot]$ is the classical exponential function and for definiteness Log is the principal value for the natural logarithmic function. This quadrature formula has the potential advantage that it should, in principle, allow for a direct comparison of $e_\lambda(t, t_0; \mathbb{T})$ with the classical exponential function, but in practice the calculation of the "integral" and its asymptotic behavior as $t \to +\infty$ could still be difficult. According to the theory of integration on time scales (see, for example, [3, p. 29]), in case $\mu(\tau) > 0$ for all $\tau \in \mathbb{T}$,

$$\int_{t_0}^{t} \frac{1}{\mu(\tau)} \mathrm{Log}(1 + \lambda\mu(t))\, \Delta\tau = \sum_{\tau \in [t_0, t)} \mathrm{Log}(1 + \lambda\mu(\tau))$$

and this amounts to the representation of $e_\lambda(t, t_0; \mathbb{T})$ as a product, which still does not immediately yield its asymptotic behavior.

In the absence of an established, general theory for the asymptotic behavior of integrals of the above type, we will investigate the behavior using several kinds of ad hoc methods on a class of examples. We emphasize that some of these examples can be treated by several of the methods and we do not claim that the method chosen for a particular example is necessarily the most direct or easiest one. The first method involves situations where there is a solution using Gamma functions and employs Stirling's formula, the second involves a direct estimation of a product representation, and the third involves using some preliminary (Ansatz) transformations and an analogue of a result of N. Levinson.

Other than the classical time scales $\mathbb{R}$ and $\mathbb{N}_0$ mentioned above and the q–difference time scale $q^{\mathbb{N}_0}$ to be discussed in Section 2.2, we will first focus our attention on classes of intermediate time scales $\mathbb{N}_0^p$, for p a positive real number. For these we have

$$\mu(n^p) = (n+1)^p - n^p = \frac{p}{n^{1-p}} + \frac{p(p-1)}{2n^{2-p}} + \dots \quad \text{for } n \geq 2, \tag{2.2}$$

or equivalently, by replacing n^p by t,

$$\mu(t) = t\left[(1+t^{-1/p})^p - 1\right] = \frac{p}{t^{1/p-1}} + \frac{p(p-1)}{2t^{2/p-1}} + \dots \quad \text{for } t > 1, t \in \mathbb{T}.$$

We will link the asymptotic growth of the exponential to the growth or decay of $\mu(t)$ as $t \to \infty$. Note that when $p < 1$, $\mu(t) \to 0$ whereas when $p > 1$, $\mu(t) \to +\infty$. Our goal is to explicitly quantify the dependence of the asymptotic behavior (as $t \to \infty$) of $\exp_\lambda(t, t_0; \mathbb{N}_0^p)$ on p. In the sequel we find it convenient to use the standard Hardy–Littlewood symbols "o" and "O" (see, for example, [6, pp. 3-10]).

We summarize some of these results in Table 1 below. As usual, we denote by $\ln(\cdot)$ the principal value of the natural logarithm with real and positive argument.

Remark 1: The constant c in Table 1 is a real number which depends upon the normalization at t_0. In (2.9), c is explicitly known (see Section 2.1). In all the other cases this "connection constant" is only known implicitly.

We start our investigation with time scales that have an explicit solution.

2.1. **Explicit solutions:** We consider here time scales of the form

$$\mathbb{N}_0^p, \quad \text{where } p = 2, 3, 4, \dots .$$

When $p = 2$, Bohner and Peterson [3, p.69] gave an explicit formula for the exponential function with $\lambda = 1$ as

$$e_1(t, 0; \mathbb{N}_0^2) = 2^{\sqrt{t}}\,(\sqrt{t})! \qquad \text{for } t \in \mathbb{T}.$$

Here we generalize that situation and then discuss the asymptotic behavior of this class of exponential functions. Using (2.2), the dynamic equation (2.1) can be written

Table 1. Asymptotic behavior of exponentials for some specific time scales

$\mathbb{T}$	$\mu(t)$	$e_\lambda(t,t_0;\mathbb{T})$	see eq.
$\mathbb{R}$	0	$e^{\lambda(t-t_0)}$	
$\ln\mathbb{N}$	$\ln(e^t+1)-t$	$ce^{\lambda t}\left[1+\mathrm{O}(e^{-t})\right]$	(2.17)
$\mathbb{N}_0^p$ $p\in(0,\frac{1}{2})$	$t\left[(1+t^{-\frac{1}{p}})^p-1\right]$	$ce^{\lambda t}\left[1+\mathrm{O}\left(\frac{1}{t^{\frac{1}{p}-2}}\right)\right]$	(2.13)
$\sqrt{\mathbb{N}_0}$	$\sqrt{t^2+1}-t$	$ce^{\lambda t}\,t^{-\frac{\lambda^2}{4}}\left[1+\mathrm{O}\left(\frac{1}{t}\right)\right]$	(2.32)
$\mathbb{N}_0^p$ $p\in(\frac{1}{2},\frac{2}{3})$	$t\left[(1+t^{-\frac{1}{p}})^p-1\right]$	$ce^{\lambda t}\,e^{-\frac{\lambda^2p^2}{2(2p-1)}t^{(2p-1)/p}}\left[1+\mathrm{O}\left(\frac{1}{t^{\frac{2}{p}-3}}\right)\right]$	(2.14)
$\mathbb{N}_0^{2/3}$	$t\left[(1+t^{-\frac{3}{2}})^{\frac{2}{3}}-1\right]$	$ce^{\lambda t}\,e^{-\frac{2\lambda^2}{3}\sqrt{t}}\,t^{4\lambda^3/27}\left[1+\mathrm{O}\left(\frac{1}{\sqrt{t}}\right)\right]$	(2.15)
$\mathbb{N}_0$	1	$(1+\lambda)^{t-t_0}$	
$\mathbb{N}_0^p$, $p=2,3,\dots$	$[(t^{1/p}+1)^p-t]$	$c(\lambda p)^{t^{1/p}}\,e^{(p-1)t^{1/p}[\ln(t^{1/p})-1]}[1+\mathrm{o}(1)]$	(2.9)
$q^{\mathbb{N}_0}$, $q>1$	$(q-1)t$	$c\{\lambda(q-1)\}^{\log_q t}\,t^{\frac{\log_q t-1}{2}}\left[1+\mathrm{O}\left(\frac{1}{t}\right)\right]$	(2.16)

as

$$y\left((n+1)^p\right)=(1+\lambda[(n+1)^p-n^p])\,y(n^p)=q(n)\,y(n^p),\quad y(n_0)=1. \tag{2.3}$$

$q(n)$ is a polynomial of degree $p-1$ in n with leading coefficient λp and therefore can be factored as

$$q(n)=\lambda p(n-\alpha_1)\,(n-\alpha_2)\,\dots\,(n-\alpha_{p-1}). \tag{2.4}$$

Here $\alpha_j\in\mathbb{C}$ for $1\le j\le p-1$. Then for $n\ge n_0$

$$\begin{aligned} e_\lambda(n^p,n_0^p;\mathbb{N}_0^p) &= \prod_{\nu=n_0}^{n-1}\lambda p(\nu-\alpha_1)(\nu-\alpha_2)\cdots(\nu-\alpha_{p-1}) \\ &= (\lambda p)^{n-n_0}\frac{\Gamma(n-\alpha_1)}{\Gamma(n_0-\alpha_1)}\frac{\Gamma(n-\alpha_2)}{\Gamma(n_0-\alpha_2)}\cdots\frac{\Gamma(n-\alpha_{p-1})}{\Gamma(n_0-\alpha_{p-1})}, \end{aligned} \tag{2.5}$$

as one sees by using the functional equation for the Gamma function. Observe that $q(n)\neq 0$ for all $n\ge n_0$ since we assumed that $1+\mu(t)\lambda\neq 0$ for all $t\in\mathbb{T}$ (note that $1+\mu(n^p)\lambda\neq 0$ for all $\lambda\in\mathbb{C}$ when n is sufficiciently large; in the special case where $\lambda>0$, this holds for all $n\ge 0$). Thus $n-\alpha_j\neq 0$ for $1\le j\le p-1$ and $n\ge n_0$. This implies that $n-\alpha_j\neq -m$, where $m\in\mathbb{N}_0$ (for $1\le j\le n$ and $n\ge n_0$) and all the gamma functions in (2.5) are well defined.

Setting $t = n^p$ and $t_0 = n_0^p$, it follows from (2.5)

$$e_\lambda(t, t_0; \mathbb{N}_0^p) = (\lambda p)^{(t^{1/p} - t_0^{1/p})} \frac{\Gamma(t^{1/p} - \alpha_1)}{\Gamma(t_0^{1/p} - \alpha_1)} \cdots \frac{\Gamma(t^{1/p} - \alpha_{p-1})}{\Gamma(t_0^{1/p} - \alpha_{p-1})}. \tag{2.6}$$

By a well-known asymptotic property of the gamma function, $\Gamma(t^{1/p} - \alpha) = \Gamma(t^{1/p})\, t^{-\alpha/p} [1 + \mathrm{o}(1)]$ as $t \to \infty$. Expanding $q(n)$ in (2.3) and comparing this with (2.4), one finds that $-(\alpha_1 + \ldots \alpha_{p-1}) = (p-1)/2$. Then (2.6) implies that

$$e_\lambda(t, t_0; \mathbb{N}_0^p) = c_0\, (\lambda p)^{t^{1/p}}\, [\Gamma(t^{1/p})]^{p-1}\, t^{\frac{p-1}{2p}}\, [1 + \mathrm{o}(1)] \qquad \text{as } t \to \infty, \tag{2.7}$$

where

$$c_0^{-1} = (\lambda p)^{t_0^{1/p}}\, \Gamma(t_0^{1/p} - \alpha_1)\, \Gamma(t_0^{1/p} - \alpha_2) \ldots \Gamma(t_0^{1/p} - \alpha_{p-1}). \tag{2.8}$$

Invoking Stirling's formula (see, e.g. [6, pp. 1 and 70]), it follows that

$$\left[\Gamma(t^{1/p})\right]^{p-1} = (2\pi)^{\frac{p-1}{2}} \frac{1}{t^{\frac{p-1}{2p}}} e^{(p-1)t^{1/p}[\ln(t^{1/p})-1]} [1 + \mathrm{o}(1)] \text{ as } t \to \infty.$$

Using this result in (2.7) yields that

$$e_\lambda(t, t_0; \mathbb{N}_0^p) = c_0\, (2\pi)^{\frac{p-1}{2}}\, (\lambda p)^{t^{1/p}}\, e^{(p-1)t^{1/p}[\ln(t^{1/p})-1]}[1 + \mathrm{o}(1)] \qquad \text{as } t \to \infty, \tag{2.9}$$

with c_0 given in (2.8).

For $h\mathbb{N}_0^p$, $p = 2, 3, 4, \ldots$ with a positive constant h, one can show similarly that

$$e_\lambda(t, t_0; h\mathbb{N}_0^p) = \tilde{c}_0\, (2\pi)^{\frac{p-1}{2}}\, (h\lambda p)^{(t/h)^{1/p}}\, e^{(p-1)(t/h)^{1/p}[\ln((t/h)^{1/p})-1]}[1 + \mathrm{o}(1)] \qquad \text{as } t \to \infty,$$

with $\tilde{c}_0^{-1} = (h\lambda p)^{(t_0/h)^{1/p}}\, \Gamma((t_0/h)^{1/p} - \alpha_1)\, \Gamma((t_0/h)^{1/p} - \alpha_2) \ldots \Gamma((t_0/h)^{1/p} - \alpha_{p-1})$.

2.2. **Product representation and direct estimation.** For time scales with $\mu(t) > 0$, we have observed above that the exponentials have explicit representations which can be expressed as ordinary products. To investigate the asymptotic behavior of these products, we look for terms which account for the gross asymptotic behavior and then estimate the modified products using standard techniques.

As a first example, we consider $\mathbb{T} = \mathbb{N}_0^p = \{0, 1, 2^p, \ldots\}$ for $0 < p < \frac{1}{2}$. Then (2.1) is equivalent to

$$y\left((n+1)^p\right) = [1 + \lambda\, \mu(n^p)] y(n^p)\,, \quad y(n_0) = 1, \tag{2.10}$$

and thus for $n \geq n_0$

$$e_\lambda(n^p, n_0^p; \mathbb{N}_0^p) = \prod_{\nu=n_0}^{n-1} [1 + \lambda\, \mu(\nu^p)]. \tag{2.11}$$

Recall that by (2.2), $\mu(\nu^p) = \mathrm{O}\left(\frac{1}{\nu^{1-p}}\right)$ as $\nu \to \infty$. In the following, fix $n_1 \geq n_0$ such that $|\lambda\mu(\nu^p)| < 1$ for all $\nu \geq n_1$. Then for n sufficiently large

$$\begin{aligned}\mathrm{Log}\,(e_\lambda(n^p, n_0^p; \mathbb{N}_0^p)) &= \sum_{\nu=n_0}^{n_1-1} \mathrm{Log}[1+\lambda\,\mu(\nu^p)] + \sum_{\nu=n_1}^{n-1} \mathrm{Log}[1+\lambda\,\mu(\nu^p)] \\ &= c_1 + \sum_{\nu=n_1}^{n-1} \left\{\lambda\mu(\nu^p) + \mathrm{O}(\mu^2(\nu^p))\right\},\end{aligned}$$

for some constant c_1. Here $\mathrm{Log}(\cdot)$ denotes the principal value of the natural logarithm. Since $\displaystyle\sum_{\nu=n_1}^{n-1} |\mathrm{O}(\mu^2(\nu^p))| \leq c_2 \sum_{\nu=n_1}^{n-1} \frac{1}{\nu^{2(1-p)}}$ for some positive constant c_2 and $0 < p < 1/2$, the sums are absolutely convergent.

For $\alpha > 1$, the integral test shows that

$$\sum_{\nu=n_1}^{n-1} \frac{1}{\nu^\alpha} = k - \sum_{\nu=n}^{\infty} \frac{1}{\nu^\alpha} = k + \mathrm{O}\left(\frac{1}{n^{\alpha-1}}\right), \tag{2.12}$$

for some positive constant k as $n \to \infty$.

Setting $\alpha = 2(1-p) > 1$, (2.12) implies that

$$\mathrm{Log}\,(e_\lambda(n^p, n_0^p; \mathbb{N}_0^p, 0 < p < 1/2)) = c_3 + \lambda n^p + \mathrm{O}\left(\frac{1}{n^{1-2p}}\right),$$

for some constant c_3 as $n \to \infty$. Finally, putting $t = n^p$ one finds that

$$e_\lambda(t, t_0; \mathbb{N}_0^p, 0 < p < 1/2) = c\, e^{\lambda t}\left[1 + \mathrm{O}\left(\frac{1}{t^{\frac{1}{p}-2}}\right)\right] \qquad t \to \infty,\ t \in \mathbb{N}_0^p,\ 0 < p < 1/2, \tag{2.13}$$

for some constant c.

An analogous computation would also yield (2.13) for the time scales $h\mathbb{N}^p$, $0 < p < 1/2$ with a positive constant h.

Note that for $p = 1/2$, $\mu^2(\sqrt{n}) = \frac{1}{4n} - \frac{1}{8n^2} + \ldots$ which is not summable. This leads to an additional term in the exponential function on $\sqrt{\mathbb{N}_0}$ as shown in Table 1 and derived in Section 2.3.

As a second example, we consider $\mathbb{T} = \mathbb{N}_0^p$ for $\frac{1}{2} < p < \frac{2}{3}$. From (2.2) and (2.11) follows that as $n \to \infty$

$$\begin{aligned}\mathrm{Log}\,(e_\lambda(n^p, n_0^p; \mathbb{N}_0^p)) &= c_1 + \sum_{\nu=n_1}^{n-1} \left\{\lambda\mu(\nu^p) - \frac{\lambda^2\mu^2(\nu^p)}{2} + \mathrm{O}(\mu^3(\nu^p))\right\} \\ &= c_1 + \lambda n^p - \lambda n_1^p - \frac{\lambda^2 p^2}{2} \sum_{\nu=n_1}^{n-1} \frac{1}{\nu^{2(1-p)}} + \sum_{\nu=n_1}^{n-1} \mathrm{O}\left(\frac{1}{\nu^{3(1-p)}}\right),\end{aligned}$$

where c_1 is some constant and n_1 is fixed such that $|\mu(\nu^p)| < 1$ for all $\nu \geq n_1$.

To compute $\displaystyle\sum_{\nu=n_1}^{n-1} \frac{1}{\nu^{2(1-p)}}$, note that

$$(\nu+1)^{2p-1} - \nu^{2p-1} = \frac{2p-1}{\nu^{2-2p}} + \mathrm{O}\left(\frac{1}{\nu^{3-2p}}\right)$$

and therefore

$$\operatorname{Log}\left(e_\lambda(n^p, n_0^p; \mathbb{N}^p)\right) = c_2 + \lambda n^p - \frac{\lambda^2 p^2}{2(2p-1)} n^{2p-1} + \sum_{\nu=n_1}^{n-1} \mathrm{O}\left(\frac{1}{\nu^{3-3p}}\right),$$

for some constant c_2 as $n \to \infty$. By (2.12) with $\alpha = 3 - 3p > 1$, it follows that

$$\sum_{\nu=n_1}^{n-1} \mathrm{O}\left(\frac{1}{\nu^{3-3p}}\right) = c_3 + \mathrm{O}\left(\frac{1}{n^{2-3p}}\right),$$

for some constant $c_3 > 0$ as $n \to \infty$. Thus putting $t = n^p$ implies that for $p \in (1/2, 2/3)$

$$e_\lambda(t, t_0; \mathbb{N}_0^p) = c\, e^{\lambda t}\, e^{-\frac{\lambda^2 p^2}{2(2p-1)} t^{(2p-1)/p}} \left[1 + \mathrm{O}\left(\frac{1}{t^{\frac{2}{p}-3}}\right)\right], \quad t \to \infty, \ t \in \mathbb{N}_0^p \tag{2.14}$$

for some constant c.

Remark 2: For $\mathbb{T} = h\mathbb{N}_0^p$ $(h > 0)$, one can show similarly that for $p \in (1/2, 2/3)$

$$e_\lambda(t, t_0; h\mathbb{N}_0^p) = c\, e^{\lambda t}\, e^{-\frac{\lambda^2 p^2 h^{1/p}}{2(2p-1)} t^{(2p-1)/p}} \left[1 + \mathrm{O}\left(\frac{1}{t^{\frac{2}{p}-3}}\right)\right], \quad t \to \infty, \ t \in h\mathbb{N}_0^p.$$

Remark 3: For $p = 2/3$, $\mu^3(n^{2/3}) = \frac{8}{27n} + \ldots$ and therefore is not summable, leading to an additional term in $e_\lambda(t, t_0; \mathbb{N}^{2/3})$. Similar computations reveal that

$$e_\lambda(t, t_0; \mathbb{N}^{2/3}) = c\, e^{\lambda t}\, e^{-\frac{2\lambda^2}{3}\sqrt{t}}\, t^{4\lambda^3/27} \left[1 + \mathrm{O}\left(\frac{1}{\sqrt{t}}\right)\right] \quad t \to \infty, \ t \in \mathbb{N}^{2/3}. \tag{2.15}$$

This product representation approach also applies to other classes of time scales. As a last example in this section, we consider a so-called q–difference equation. Here $\mathbb{T} = q^{\mathbb{N}_0} = \{1, q, q^2, \ldots\}$, where we assume that $q > 1$. The graininess is given by $\mu(t) = (q-1)t$ being unbounded as $t \to \infty$. Then (2.1) is equivalent to

$$y(qt) = [1 + \lambda(q-1)t]\, y(t), \quad y(q^{n_0}) = 1.$$

Setting $t = q^n$ and introducing the abbreviation $\gamma = \lambda(q-1)$ we have for $n \ge n_0$

$$e_\lambda(q^n, q^{n_0}; q^{\mathbb{N}_0}) = \prod_{\nu=n_0}^{n-1} \gamma q^\nu \left[1 + \frac{1}{\gamma q^\nu}\right] = \gamma^{n-n_0} \left[q^{\frac{(n-1)n}{2} - \frac{(n_0-1)n_0}{2}}\right] \prod_{\nu=n_0}^{n-1} \left[1 + \frac{1}{\gamma q^\nu}\right].$$

Using logarithm and the geometric series, one finds that

$$\operatorname{Log} \prod_{\nu=n_0}^{n-1} \left[1 + \frac{1}{\gamma q^\nu}\right] = \sum_{\nu=n_0}^{n-1} \operatorname{Log}\left[1 + \frac{1}{\gamma q^\nu}\right] = \sum_{\nu=n_0}^{n-1} \mathrm{O}\left(\frac{1}{q^\nu}\right) = c_1 + \mathrm{O}\left(\frac{1}{q^n}\right),$$

for some constant c_1 as $n \to \infty$. Here we assumed that n_0 was sufficiently large such that $\left|\frac{1}{\gamma q^\nu}\right| < 1$ for all $\nu \geq n_0$. Otherwise, split the sum up at an integer n_1 sufficiently large and proceed as in the first two examples in this section. Therefore

$$e_\lambda(q^n, q^{n_0}; q^{\mathbb{N}_0}) = c\,\gamma^n\, q^{\frac{(n-1)n}{2}} \left[1 + \mathrm{O}\left(\frac{1}{q^n}\right)\right],$$

for some constant c as $n \to \infty$. Going back to t and using $n = \log_q t$ one sees that

$$\begin{aligned} e_\lambda(t, t_0; q^{\mathbb{N}_0}) &= c\,\gamma^{\log_q t}\, \frac{q^{\frac{(\log_q t)\log_q t}{2}}}{q^{\frac{\log_q t}{2}}} \left[1 + \mathrm{O}\left(\frac{1}{t}\right)\right] \\ &= c\,\{\lambda(q-1)\}^{\log_q t}\, t^{\frac{\log_q t - 1}{2}} \left[1 + \mathrm{O}\left(\frac{1}{t}\right)\right] \qquad (2.16) \end{aligned}$$

for some constant c as as $t \to \infty$, $t \in q^{\mathbb{N}_0}$. This result may also be obtained using the results in the next section.

Remark 4: The techniques of this section also apply to the time scale $\mathbb{T} = \ln \mathbb{N}$. Here $\mu(t) = e^{-t} + \mathrm{O}\left(e^{-2t}\right)$. As this rapidly decreasing graininess might suggest,

$$e_\lambda(t, t_0; \ln \mathbb{N}) = c\, e^{\lambda t} \left[1 + \mathrm{O}\left(e^{-t}\right)\right]. \qquad (2.17)$$

2.3. **Dynamic equations approach.** In this section we use the functional equation directly and forget about a product representation for the solution. To use this method one first inspects the equation to detect terms contributing to the main part of the asymptotic and using a preliminary (often an "Ansatz") transformation, one derives a new equation with those terms suppressed. After several (if necessary) steps of this type, we obtain an equation which is a small perturbation of one with constant solutions. If the perturbation is sufficiently small, the solutions can be shown to have constant limiting behavior. To show this last step one can use either a one-dimensional analogue of a result of N. Levinson or often a more direct estimation.

As a first example, we consider $\mathbb{T} = h\sqrt{\mathbb{N}}$. Here

$$\mu(h\sqrt{n}) = \frac{h}{2\sqrt{n}} + \mathrm{O}\left(\frac{1}{n^{3/2}}\right) \quad \text{or} \quad \mu(t) = \frac{h^2}{2t} + \mathrm{O}\left(\frac{1}{t^3}\right). \qquad (2.18)$$

From the previous section, we expect the dominant term in $e_\lambda(t, t_0; h\sqrt{N_0})$ to be $e^{\lambda t}$ which suggests the substitution

$$y(t) = e^{\lambda t} z(t). \qquad (2.19)$$

Then using the product rule for time scales (see, e.g. [3, Thm. 1.20]), (1.1) becomes

$$z^\Delta(t) = \frac{[1 + \lambda\mu(t)]\, e^{-\lambda\mu(t)} - 1}{\mu(t)}\, z(t).$$

Expanding the exponential function leads to

$$z^\Delta(t) = \left\{-\lambda^2\mu(t)/2 + \mathrm{O}\left(\mu^2(t)\right)\right\}\, z(t). \qquad (2.20)$$

Our goal is to apply Levinson's Perturbation Lemma for time scales [2, Thm. 4.1] which is concerned with dynamic equations of the form

$$x^{\Delta}(t) = [\Lambda(t) + R(t)]\, x(t), \tag{2.21}$$

where $\Lambda(t)$ and $R(t)$ are $m \times m$ matrices and $\Lambda(t) = \operatorname{diag}\{\lambda_1(t), \ldots, \lambda_m(t)\}$ is diagonal. Depending on a certain dichotomy condition on the entries of $\Lambda(t)$ and a growth condition of $R(t)$ with respect to $\Lambda(t)$ of the form

$$\int_{t_0}^{\infty} \left| \frac{R(t)}{1+\mu(t)\lambda_i(t)} \right| \Delta t < \infty, \quad 1 \le i \le m, \tag{2.22}$$

this Perturbation Lemma implies that (2.21) has a fundamental matrix X such that

$$X(t) = [I + \mathrm{o}(1)]Y(t) \quad \text{as } t \to \infty, \tag{2.23}$$

where Y satisfies $Y^{\Delta}(t) = \Lambda(t)Y(t)$ and $Y(t_0) = I$.

We will use Levinson in the one-dimensonal case ($m = 1$) with $\Lambda(t) = 0$ for all t. Then the dichotomy condition on $\Lambda(t)$ is trivially satisfied and (2.22) reduces to

$$\int_{t_0}^{\infty} |R(t)|\; \Delta t < \infty. \tag{2.24}$$

In this one-dimensional case, one can also show that (2.23) can be refined to

$$X(t) = \left[I + \mathrm{O}\left(\int_t^{\infty} |R(\tau)|\, \Delta\tau \right) \right] Y(t) \quad \text{as } t \to \infty. \tag{2.25}$$

We will discuss this and other results on the asymptotic behavior of perturbed dynamical equations in a subsequent paper.

Viewing (2.20) now as being of the form $z^{\Delta}(t) = \left[0 + \tilde{R}(t)\right] z(t)$, it follows from (2.18)

$$\int_{t_0}^{\infty} \left|\tilde{R}(t)\right|\; \Delta t = \sum_{n=n_0}^{\infty} |\tilde{R}(h\sqrt{n})|\, \mu(h\sqrt{n}) = \sum_{n=n_0}^{\infty} \left\{ -\frac{\lambda^2 h^2}{8n} + \mathrm{O}\left(\frac{1}{n^{3/2}} \right) \right\},$$

which does not satisfy (2.24).

To eliminate the $1/n$–term, we try an Ansatz

$$z(t) = t^{\beta} w(t) \quad \text{with } \beta \text{ to be chosen appropriately.} \tag{2.26}$$

Then

$$w^{\Delta}(t) = \left[\left\{ -\lambda^2\mu^2(t)/2 + \mathrm{O}(\mu^3(t)) + 1 \right\} \left\{ \frac{t}{\sigma(t)} \right\}^{\beta} - 1 \right] \frac{w(t)}{\mu(t)}. \tag{2.27}$$

Now for t sufficiently large such that $|\mu(t)/t| < 1$,

$$\left(\frac{t}{\sigma(t)} \right)^{\beta} = \left(1 + \frac{\mu(t)}{t} \right)^{-\beta} = 1 - \beta\frac{\mu(t)}{t} + \mathrm{O}\left(\frac{\mu^2(t)}{t^2} \right), \tag{2.28}$$

and therefore

$$w^{\Delta}(t) = \left[\left\{1 - \frac{\lambda^2 h^4}{8t^2} + \mathrm{O}\left(\frac{1}{t^3}\right)\right\}\left\{1 - \frac{\beta h^2}{2t^2} + \mathrm{O}\left(\frac{1}{t^4}\right)\right\} - 1\right]\frac{w(t)}{\mu(t)}. \tag{2.29}$$

Choosing $\beta = -\lambda^2 h^2/4$ implies that

$$w^{\Delta}(t) = \mathrm{O}\left(\frac{1}{t^3}\right)\frac{w(t)}{\mu(t)} = [0 + R(t)]\, w(t), \tag{2.30}$$

where now $R(t)$ satisfies (2.24) and, moreover, $\int_t^\infty |R(\tau)|\,\Delta\tau = \mathrm{O}\left(\frac{1}{t}\right)$. By Levinson's Perturbation Lemma and (2.25), (2.29) has, for $t \in h\sqrt{\mathbb{N}}$, a solution

$$w(t) = 1 + \mathrm{O}\left(\frac{1}{t}\right) \quad \text{as } t \to \infty. \tag{2.31}$$

For any given time scale, the quotient of any two non-trivial solutions of (1.1) satisfies the trivial dynamic equation $u^{\Delta}(t) = 0$ implying that the solutions are constant multiples of each other. Therefore the solution of the IVP (2.1) is given by

$$e_\lambda(t, t_0; h\sqrt{\mathbb{N}_0}) = c\, e^{\lambda t} t^{-\lambda^2 h^2/4}\left[1 + \mathrm{O}\left(\frac{1}{t}\right)\right] \qquad t \to \infty, \ \ t \in h\sqrt{\mathbb{N}}, \tag{2.32}$$

for some constant c.

Remark 5: (2.31) could also have been derived by rewriting (2.30) as

$$w(h\sqrt{n+1}) = \left[1 + \mathrm{O}\left(\frac{1}{n^{3/2}}\right)\right] w(h\sqrt{n})$$

and using that for $\alpha > 1$, there exists a constant γ such that

$$\prod_{\nu=n_0}^{n-1}\left[1 + \mathrm{O}\left(\frac{1}{\nu^\alpha}\right)\right] = \gamma + \mathrm{O}\left(\frac{1}{n^{\alpha-1}}\right) \quad \text{as } n \to \infty, n \in \mathbb{N}.$$

This follows from invoking the logarithm and its expansion (for n sufficiently large) and utilizing (2.12).

Remark 6: Note that for any given constant $\alpha > 0$, this example permits to find time scales that have an exponential function $e_\lambda(t, t_0; \mathbb{T})$ with asymptotic behavior $c\, e^{\lambda t} t^{-\lambda^2 \alpha}\,[1 + \mathrm{o}(1)]$ as $t \to \infty$. An example would be $\mathbb{T} = 2\sqrt{\alpha\,\mathbb{N}}$ (see (2.32)).

The dynamical equations approach also applies to some implicitly defined time scales for which only the forward jump operator $\sigma(t) > t$ is known, not the specific points in $\mathbb{T}$. Then the time scale is given as the set $\mathbb{T} = \{t_0, \sigma(t_0), \sigma(\sigma(t_0)), \ldots\}$ for some $t_0 \in \mathbb{R}$.

As a preliminary, we state the following proposition that can be found in a slightly different form in [9, Thm. 1.3.5]. This proposition provides an estimate for the points on the time scale thereby allowing us to check if a perturbation $R(t)$ satisfies Levinson's growth condition. In the following, ρ denotes a real parameter, not the backward jump operator.

Proposition 2.1. For $t \geq t_0$ ($t_0 \in \mathbb{R}$) consider

$$\sigma(t) = t + \mu(t) = t\left\{1 + \frac{\kappa}{t^\rho} + \mathrm{O}\left(\frac{1}{t^\gamma}\right)\right\} \text{ as } t \to \infty, \tag{2.33}$$

where $\gamma > \rho \geq 1$ and $\kappa > 0$. Assume that t_0 is sufficiently large such that $\mu(t) > 0$ for all $t \geq t_0$. Define the iterates of σ by $\sigma^1(t) = \sigma(t)$ and $\sigma^{n+1}(t) = \sigma(\sigma^n(t))$. Then

$$\lim_{n\to\infty} \frac{\sigma^n(t)}{n^{1/\rho}} = (\rho\kappa)^{1/\rho} \quad \text{ for all } t \geq t_0. \tag{2.34}$$

Proof. The proof is a modification of the proof of [9, Thm. 1.3.5]. Observe that

$$\sigma(t^{1/\rho}) = t^{1/\rho}\left(1 + \frac{\kappa}{t} + \mathrm{O}\left(\frac{1}{t^{\gamma/\rho}}\right)\right) \quad \text{as } t \to \infty.$$

To simplify the iterations, we use the auxiliary function

$$\varphi(t) = \left(\sigma(t^{1/\rho})\right)^\rho = t\left(1 + \frac{\kappa}{t} + \mathrm{O}\left(\frac{1}{t^{\gamma/\rho}}\right)\right)^\rho = t + \rho\kappa + \mathrm{O}\left(\frac{1}{t^\delta}\right) \quad \text{as } t \to \infty,$$

where $\delta = \min\{1, \gamma/\rho - 1\}$. An inductive argument shows that the nth iterate of $\varphi(t)$ satisfies for $n \geq 1$

$$\varphi^n(t) = t + n\rho\kappa + \mathrm{O}\left(\frac{1}{t^\delta}\right) = n\rho\kappa\left[1 + \frac{1}{n\rho\kappa}\left\{t + \mathrm{O}\left(\frac{1}{t^\delta}\right)\right\}\right] \quad \text{as } t \to \infty, \tag{2.35}$$

where $\mathrm{O}\left(\frac{1}{t^\delta}\right)$ is bounded in both n and t. This follows from

$$\begin{aligned}\varphi^{n+1}(t) &= \varphi(\varphi^n(t)) = t + n\rho\kappa + \mathrm{O}\left(\frac{1}{t^\delta}\right) + \rho\kappa + \mathrm{O}\left(\frac{1}{(\varphi^n(t))^\delta}\right)\\ &= t + (n+1)\rho\kappa + \mathrm{O}\left(\frac{1}{t^\delta}\right) + \mathrm{O}\left(\frac{1}{t^\delta\left\{1 + \frac{n\rho\kappa}{t} + \mathrm{O}(\frac{1}{t^{1+\delta}})\right\}^\delta}\right) \quad \text{as } t \to \infty.\end{aligned}$$

Note that the very last term is also of order $\mathrm{O}\left(\frac{1}{t^\delta}\right)$ as $t \to \infty$ both in n and t. Since from (2.35)

$$\lim_{n\to\infty} \frac{\varphi^n(t^\rho)}{n\rho\kappa} = 1,$$

we obtain that

$$\lim_{n\to\infty} \frac{\sigma^n(t)}{(n\rho\kappa)^{1/\rho}} = 1 \text{ for } t \geq t_0.$$

The proof is complete. □

We now consider four classes of graininess functions $\mu(t)$ corresponding to the values $\rho > 2$, $\rho = 2$, $3/2 < \rho < 2$ and $\rho = 1$ in (2.33). Table 2 below displays the asymptotic dependence of $e_\lambda(t, t_0; \sigma(t))$ on the parameters in the classes.

As a first example we consider the case where the time scale is defined by the forward jump operator given in (2.33) with $\rho > 2$. Fix t_0 sufficiently large such that the graininess which is given by

$$\mu(t) = \frac{\kappa}{t^{\rho-1}} + \mathrm{O}\left(\frac{1}{t^{\gamma-1}}\right) \quad \text{as } t \to \infty,$$

is positive for $t \geq t_0$, $t \in \mathbb{T}$. Let $t_n := \sigma^n(t_0)$ denote the "nth point" in the time scale.

TABLE 2. Asymptotic behavior of exponentials for some classes of time scales

$\mu(t)$	$e_\lambda(t,t_0;\sigma(t))$	see eq.
$\frac{\kappa}{t^{\rho-1}}+\mathrm{O}\left(\frac{1}{t^{\gamma-1}}\right)$ $\gamma>\rho>2,\ \ \kappa>0$	$c\,e^{\lambda t}\left[1+\mathrm{O}\left(\frac{1}{t^{\rho-2}}\right)\right]$	(2.37)
$\frac{\kappa}{t}+\mathrm{O}\left(\frac{1}{t^{\gamma-1}}\right)$ $\gamma>2,\ \ \kappa>0$	$c\,e^{\lambda t}t^{-\frac{\lambda^2\kappa}{2}}\left[1+\mathrm{O}\left(\frac{1}{t^{\delta-2}}\right)\right]$ $\delta=\min\{\gamma,3\}$	(2.41)
$\frac{\kappa}{t^{\rho-1}}+\mathrm{O}\left(\frac{1}{t^{\gamma-1}}\right)$ $\rho\in(\frac{3}{2},2)\,,\ \gamma>2$	$c\,e^{\lambda t}e^{-\frac{\lambda^2\kappa}{2(2-\rho)}t^{2-\rho}}\,[1+\mathrm{o}(1)]$	(2.45)
$\kappa+\frac{c_1}{t}+\mathrm{O}\left(\frac{1}{t^{\delta}}\right)$ $\delta>1\,,\ 1+\kappa\lambda>0$	$c\,(1+\kappa\lambda)^{t/\kappa}\,t^{\beta}\,\left[1+\mathrm{O}\left(\frac{1}{t^{\eta-1}}\right)\right]$ $\eta=\min\{\delta,2\}\,,\ \beta=\frac{\lambda c_1}{(1+\kappa\lambda)\kappa}-\frac{c_1\ln(1+\kappa\lambda)}{\kappa^2}$	(2.44)

Then (1.1) and (2.19) lead to (2.20) which we consider as being in Levinson form $z^\Delta(t)=[0+R(t)]z(t)$, with $R(t)=\mathrm{O}\left(\mu(t)\right)$. By Proposition 2.1,

$$t_n=[\rho\kappa n]^{1/\rho}\,[1+\mathrm{o}(1)]\quad\text{as } n\to\infty. \tag{2.36}$$

Thus

$$R(t_n)\mu(t_n)=\mathrm{O}\left(\mu^2(t_n)\right)=\mathrm{O}\left(\frac{1}{t_n^{2(\rho-1)}}\right)=\mathrm{O}\left(\frac{1}{n^{2(\rho-1)/\rho}}\right)$$

and therefore

$$\int_{t_0}^{\infty}|R(t)|\,\Delta t=\sum_{n=0}^{\infty}|R(t_n)|\mu(t_n)<\infty,$$

by the hypothesis that $\rho>2$. A straightforward computation reveals that

$$\int_t^{\infty}|R(\tau)|\,\Delta\tau=\mathrm{O}\left(\frac{1}{t^{\rho-2}}\right),$$

and therefore (2.25) implies that

$$e_\lambda(t,t_0;\sigma(t))=c\,e^{\lambda t}\left[1+\mathrm{O}\left(\frac{1}{t^{\rho-2}}\right)\right]. \tag{2.37}$$

As a second example, we consider the time scale characterized by the forward jump operator given in (2.33) with $\rho=2$ or, equivalently,

$$\mu(t)=\frac{\kappa}{t}+\mathrm{O}\left(\frac{1}{t^{\gamma-1}}\right)\qquad(t\to\infty) \tag{2.38}$$

for some $\gamma>2$. For $t_0\in\mathbb{T}$ sufficintly large such that $\mu(t)>0$ for all $t\geq t_0$, we make again the preliminary transformation (2.19) in (1.1) and arrive at (2.20). Viewing this again as $z^\Delta(t) = [0 + \tilde{R}(t)]\, z(t)$, Proposition 2.1 implies in this example that the nth points t_n on the time scale are characterized by

$$t_n = \sqrt{2\kappa n}\,[1 + \mathrm{o}(1)] \quad \text{as } n \to \infty, \tag{2.39}$$

and therefore

$$\int_{t_0}^{\infty} |\tilde{R}(t)|\,\Delta t = \sum_{n=n_0}^{\infty} |\tilde{R}(t_n)|\mu(t_n) = -\frac{\lambda^2\kappa^2}{2}\sum_{n=n_0}^{\infty}\frac{1}{t_n^2} + \dots$$

diverges. To eliminate the $1/t_n^2$-term, we tentatively make a preliminary transformation of the form (2.26). Then (2.27) and (2.28) lead to

$$w^\Delta(t) = \left[-\frac{\lambda^2\mu^2(t)}{2} - \frac{\beta\mu(t)}{t} + \mathrm{O}(\mu^3(t)) + \mathrm{O}\left(\frac{\mu^2(t)}{t^2}\right)\right]\frac{w(t)}{\mu(t)}.$$

We can rewrite this using (2.38) and find

$$w^\Delta(t) = \left[-\frac{\lambda^2\kappa^2}{2t^2} - \frac{\beta\kappa}{t^2} + \mathrm{O}\left(\frac{1}{t^\delta}\right)\right]\frac{w(t)}{\mu(t)},$$

where $\delta = \min\{\gamma, 3\} > 2$. Choosing $\beta = -\lambda^2\kappa/2$ implies

$$w^\Delta(t) = \mathrm{O}\left(\frac{1}{t^\delta}\right)\frac{w(t)}{\mu(t)} = [0 + R(t)]\, w(t), \tag{2.40}$$

where $R(t)$ is now absolutely integrable by (2.39). Since

$$\int_t^{\infty} |R(\tau)|\,\Delta\tau = \mathrm{O}\left(\frac{1}{t^{\delta-2}}\right),$$

(2.25) now implies that (2.40) has a solution $w(t) = 1 + \mathrm{O}\left(\frac{1}{t^{\delta-2}}\right)$ as $t \to \infty$. Therefore the IVP (2.1) has in this case the solution

$$e_\lambda(t, t_0; \sigma(t)) = c\, e^{\lambda t} t^{-\frac{\lambda^2\kappa}{2}}\left[1 + \mathrm{O}\left(\frac{1}{t^{\delta-2}}\right)\right]. \tag{2.41}$$

Remark 7: Looking backwards, our first example in this section ($\mathbb{T} = h\sqrt{N_0}$) is just a special case of time scales given by (2.38) with $\kappa = h^2/2$ and $\gamma = 4$.

As a final example, we consider time scales where the forward jump operator given in (2.33) has the form

$$\sigma(t) = t + \kappa + \frac{c_1}{t} + \mathrm{O}\left(\frac{1}{t^\delta}\right) \text{ for some } \delta > 1.$$

As usual, we consider $t \geq t_0$ sufficiently large such that $\mu(t) > 0$ for all $t \geq t_0$. We also assume that $\alpha := 1 + \kappa\lambda > 0$. We note that the graininess has a constant term and make the preliminary transformation

$$y(t) = \alpha^{t/\kappa}\, z(t). \tag{2.42}$$

A motivation for this transformation is the time scale $\mathbb{T} = h\mathbb{N}_0$ which has graininess $\mu(t) = h$ and $e_\lambda(t, 0; h\mathbb{N}_0) = (1 + h\lambda)^{t/h}$. Then (1.1) implies that

$$\begin{aligned} z^\Delta(t) &= \left\{\alpha^{-\mu(t)/\kappa}\left[1 + \lambda\mu(t)\right] - 1\right\} \frac{z(t)}{\mu(t)} = \left\{e^{-\mu(t)\ln\alpha/\kappa}\left[1 + \lambda\mu(t)\right] - 1\right\} \frac{z(t)}{\mu(t)} \\ &= \left\{\frac{1}{\alpha}\left[1 - \frac{c_1 \ln\alpha}{\kappa t} + \mathrm{O}\left(\frac{1}{t^\eta}\right)\right]\left[\alpha + \frac{\lambda c_1}{t} + \mathrm{O}\left(\frac{1}{t^\delta}\right)\right] - 1\right\} \frac{z(t)}{\mu(t)} \\ &= \left\{\frac{c_1}{t}\left(\frac{\lambda}{\alpha} - \frac{\ln\alpha}{\kappa}\right) + \mathrm{O}\left(\frac{1}{t^\eta}\right)\right\} \frac{z(t)}{\mu(t)}, \end{aligned}$$

where $\eta = \min\{\delta, 2\} > 1$. Viewing this as a perturbed equation $z^\Delta(t) = [0+R(t)]\, z(t)$, Proposition 2.1 yields for the nth points t_n on the time scale that $t_n = n\kappa[1 + \mathrm{o}(1)]$ as $n \to \infty$ and thus $\int_{t_0}^\infty |R(t)|\,\Delta t$ diverges. Therefore we make another preliminary transformation of the form

$$z(t) = t^\beta w(t),$$

with β to be determined. This leads to

$$w^\Delta(t) = \frac{\left(\frac{t}{\sigma(t)}\right)^\beta \left[1 + \frac{c_1}{t}\left(\frac{\lambda}{\alpha} - \frac{\ln\alpha}{\kappa t}\right) + \mathrm{O}\left(\frac{1}{t^\eta}\right)\right] - 1}{\mu(t)}\, w(t).$$

By (2.28), $\left(\dfrac{t}{\sigma(t)}\right)^\beta = 1 - \dfrac{\beta\kappa}{t} + \mathrm{O}\left(\dfrac{1}{t^2}\right)$ and therefore

$$w^\Delta(t) = \left\{\frac{1}{t}\left(c_1\left[\frac{\lambda}{\alpha} - \frac{\ln\alpha}{\kappa}\right] - \beta\kappa\right) + \mathrm{O}\left(\frac{1}{t^\eta}\right)\right\} \frac{w(t)}{\mu(t)}.$$

Setting

$$\beta = \frac{\lambda c_1}{\alpha\kappa} - \frac{c_1 \ln\alpha}{\kappa^2} \tag{2.43}$$

implies that

$$w^\Delta = \frac{\mathrm{O}(t^{-\eta})}{\mu(t)}\, w(t).$$

Levinson's Perturbation Lemma and (2.25) now imply that

$$w(t) = 1 + O\left(\frac{1}{t^{\eta-1}}\right) \qquad \text{as } t \to \infty,\ t \in \mathbb{T}.$$

Thus we found that (with β given by (2.43), $\alpha = 1 + \kappa\lambda > 0$ and $\eta = \min\{\delta, 2\} > 1$)

$$e_\lambda(t, t_0; \mathbb{T}) = c\,(1 + \kappa\lambda)^{t/\kappa}\, t^\beta \left[1 + \mathrm{O}\left(\frac{1}{t^{\eta-1}}\right)\right]. \tag{2.44}$$

We wish to emphasize that $c_1 = 0$ implies that $\beta = 0$ and in this case

$$e_\lambda(t, t_0; \mathbb{T}) = c\,(1 + \kappa\lambda)^{t/\kappa} \left[1 + \mathrm{O}\left(\frac{1}{t^{\eta-1}}\right)\right].$$

Remark 8: One can show similarly that for time scales with $\sigma(t)$ given by (2.33), where $3/2 < \rho < 2$ and $\gamma > 2$,

$$e_\lambda(t, t_0; \mathbb{T}) = c\, e^{\lambda t} e^{-\frac{\lambda^2 \kappa}{2(2-\rho)} t^{2-\rho}} \, [1 + \mathrm{o}(1)]. \tag{2.45}$$

Remark 9: We mention that the asymptotic behavior of the exponential function in the "implicit" examples above could also have been obtained using results from the theory of functional equations (see [9, Thm. 3.3.4]). This approach would require preliminary transformations to allow applying their results to our setting plus our preliminary changes of the dependent variable to reduce to Levinson form.

2.4. **Asymptotic exponential classes.** One sees from Table 2 that for all time scales $\mathbb{T}$ having graininess

$$\mu(t) = \frac{\kappa}{t^{\rho-1}} + \mathrm{O}\left(\frac{1}{t^{\gamma-1}}\right) \text{ as } t \to +\infty$$

with any $\kappa > 0$, and $\gamma > \rho > 2$, the corresponding exponentials have exactly the same rate of growth as the corresponding ordinary exponential, up to a multiplicative factor $c + \mathrm{o}(1)$ as $t \to \infty$. Such time scales we would say are in the *same asymptotic exponential class* as $\mathbb{R}$.

While the exponential functions and the corresponding estimates in Table 2 are valid only for t in $\mathbb{T}$, we remark that since the form is independent of t_0 and by varying t_0 we can shift the time scale so that any prescribed point t would be in, we will take the point of view that the estimate holds for all t in $\mathbb{R}$, sufficiently large. Therefore in this sense we may asymptotically compare functions on two time scales $\mathbb{T}$ and $\tilde{\mathbb{T}}$ even though they may have no points in common.

We extend this idea to other classes of time scales which have exponentials having (for fixed λ) an asymptotic representation of the form

$$\exp[q(t)]\, t^m (\ln t)^n,$$

where $q(t)$ is a polynomial in real powers of t and $\ln(t)$, m is a real number, and n is a non-negative integer. Such functions are said to be in a *log-exponential class* and according to the calculations above, all time scales we have considered have exponentials lying in such a log-exponential class. Then any two such time scales are in the same asymptotic exponential class if the exponentials agree up to a factor $c + \mathrm{o}(1)$ to a particular log-exponential function and any particular time scale having such an exponential could be called a *representative for that exponential class.* Naturally, one would like to select in each class the "simplest" representative, which could mean either that the graininess is simple or that the general element in $\mathbb{T}$ has as simple a closed form as possible.

One may now re-formulate the contents of Table 2 in the following

Theorem 2.2. 1. *All time scales with* $\mu(t)$ *of the form* $\mu(t) = \frac{\kappa}{t^{\rho-1}} + \mathrm{O}\left(\frac{1}{t^{\gamma-1}}\right)$, $\kappa > 0$, $\gamma > \rho > 2$ *are in the same asymptotic exponential class* $e^{\lambda t}$ *and a representative for the class is* $\mathbb{R}$.

2. *All time scales with* $\mu(t)$ *of the form* $\mu(t) = \frac{\kappa}{t} + \mathrm{O}\left(\frac{1}{t^{1+\delta}}\right)$, $\kappa > 0$, $\delta > 0$ *are in the same asymptotic exponential class* $e^{\lambda t} t^{-\frac{\lambda^2 \kappa}{2}}$ *and a representative for the class is* $\sqrt{2\kappa \mathbb{N}_0}$.

3. *All time scales with* $\mu(t)$ *of the form* $\mu(t) = \frac{\kappa}{t^{\rho-1}} + \mathrm{O}\left(\frac{1}{t^{1+\delta}}\right)$, $\kappa > 0$, $3/2 < \rho < 2$, $\delta > 0$ *are in the same asymptotic exponential class* $e^{\lambda t} e^{-\frac{\lambda^2 \kappa}{2(2-\rho)} t^{2-\rho}}$ *and a representative for the class is* $(\rho \kappa \mathbb{N}_0)^{1/\rho}$.

4. *All time scales with* $\mu(t)$ *of the form* $c_0 + \mathrm{O}\left(\frac{1}{t^{1+\delta}}\right)$, $\delta > 0$ *and* $1 + \lambda c_0 > 0$ *are in the same asymptotic exponential class* $(1 + c_0 \lambda)^{t/c_o}$ *and a representative for the class is* $c_0 \mathbb{N}_0$.

3. MATRIX EXPONENTIAL FUNCTIONS

An $n \times n$ matrix–valued function A on a time scale $\mathbb{T}$ is *regressive* (with respect to $\mathbb{T}$) provided

$$I + \mu(t) A(t) \qquad \text{is invertible for all } t \in \mathbb{T}, \tag{3.1}$$

and we denote the class of all such regressive and rd-continuous functions by $\mathcal{R}$. By the Existence and Uniqueness Theorem for dynamic equations on time scales [3, Thm. 5.8], the initial value problem $y^\Delta = A(t)\, y$, $y(t_0) = y_0$ has a unique solution for $A \in \mathcal{R}$ and $t_0 \in \mathbb{T}$.

Because of possible non-commutativity between $A(t)$ and $\int_{t_0}^{t} A(\tau)\, \Delta\tau$, it is non-trivial to express a fundamental solution as a matrix exponential in general. But in the case of constant coefficients

$$y^\Delta(t) = Ay(t),$$

when A is in $\mathcal{R}$, one may write

$$Y(t) = P\, [\exp_J(t, t_0; \mathbb{T})] P^{-1},$$

where $P^{-1} AP = J$ is a Jordan canonical form for A. This reduces the calculation to one for each of the individual Jordan blocks. As an alternative to this procedure, other methods have been proposed which do not rely on an a priori reduction to Jordan form. One possibility is the Putzer algorithm, which we include for the convenience of the reader. This algorithm was mentioned in [1] and can also be found in [3].

3.1. **Putzer Algorithm.**

Theorem 3.1 (Putzer Algorithm). *Let $A \in \mathcal{R}$ be a constant $n \times n$ matrix. Suppose $t_0 \in \mathbb{T}$. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A, then*

$$e_A(t, t_0; \mathbb{T}) = \sum_{k=0}^{n-1} r_{k+1}(t) P_i,$$

where $r(t) := (r_1(t), r_2(t) \ \ldots, r_n(t))^T$ is the solution of the IVP

$$r^\Delta = \begin{pmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 1 & \lambda_2 & 0 & \ldots & 0 \\ 0 & 1 & \lambda_3 & & 0 \\ \vdots & & & & 0 \\ 0 & \ldots & 0 & 1 & \lambda_n \end{pmatrix} r, \qquad r(t_0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{3.2}$$

and the matrices $P_0, P_1, \ldots P_n$ are defined recursively by $P_0 = I$ and

$$P_{k+1} = (A - \lambda_{k+1} I) P_k \quad \text{for} \quad 0 \le k \le n-1.$$

3.2. **Harris Algorithm.** We now present an algorithm to compute the matrix exponential $e_A(t, t_0; \mathbb{T})$ for constant matrices $A \in \mathcal{R}$. It was introduced recently by William A. Harris Jr. in the case of ordinary differential equations and was published by Harris, Fillmore and Smith [7]. It includes the Putzer algorithm as a special case as we will show.

We use the symbol $D(f) = f^\Delta(t)$. Then, for constant matrices $A \in \mathcal{R}$, it follows by iteration that

$$D^k e_A(t, t_0; \mathbb{T}) = A^k e_A(t, t_0; \mathbb{T}). \tag{3.3}$$

Let $c(\lambda)$ denote the characteristic polynomial

$$c(\lambda) = \det(\lambda I - A) = (\lambda - \lambda_1) \ldots (\lambda - \lambda_n).$$

The Hamilton-Cayley theorem asserts that $c(A) = 0$ and therefore

$$C(D) e_A(t, t_0; \mathbb{T}) = c(A) e_A(t, t_0; \mathbb{T}) = 0.$$

Thus each entry $u_{ij}(t, t_0)$ of $e_A(t, t_0; \mathbb{T})$ is a solution of the n-th order scalar equation with constant coefficients

$$c(D) y(t) = 0. \tag{3.4}$$

Note that $I + \mu(t)A$ invertible is equivalent to $1 + \mu(t)\lambda_j \ne 0$ $(1 \le j \le n)$.

Let $W = W(y_1, \ldots, y_n)(t)$ denote the Wronski determinant of the set $\{y_1, \ldots, y_n\}$ of $(n-1)$ times delta differentiable functions, i.e., $W = \det V[y_1, \ldots, y_n]$, where

$$V[y_1, \ldots, y_n] = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1^{\Delta} & y_2^{\Delta} & \cdots & y_n^{\Delta} \\ \vdots & \vdots & & \vdots \\ y_1^{\Delta^{n-1}} & y_2^{\Delta^{n-1}} & \cdots & y_n^{\Delta^{n-1}} \end{pmatrix}.$$

Now let $y_1, \ldots y_n$ be solutions of (3.4) such that $W(y_1, \ldots y_n)(t_1) \neq 0$ for some $t_1 \in \mathbb{T}$. Then (see, e.g. [3, Thm. 5.103]) each entry u of $e_A(t, t_0; \mathbb{T})$ is a unique linear combination of $y_1, \ldots y_n$ and therefore

$$e_A(t, t_0; \mathbb{T}) = y_1(t)\, F_1 + \ldots + y_n(t)\, F_n \tag{3.5}$$

for some unique constant $n \times n$ matrices F_i $(1 \leq i \leq n)$. Invoking (3.3) and setting $t = t_0$, this implies for $i \geq 0$ that

$$A^i = (\Delta^i\, y_1)(t_0) F_1 + \ldots + (\Delta^i\, y_n)(t_0) F_n.$$

In matrix representation, this leads to

$$V[y_1, \ldots, y_n]\,(t_0) \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{pmatrix} = \begin{pmatrix} I \\ A \\ \vdots \\ A^{n-1} \end{pmatrix}.$$

Here and in what follows, the expression $\begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix}$ is not interpreted as an $n^2 \times n$ matrix, but rather as an $n \times 1$ blocked array with ith entry being F_i (for $i = 1, 2, \cdots n$). Then the matrices F_i $(1 \leq i \leq n)$ are determined by

$$\begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{pmatrix} = V^{-1}[y_1, \cdots y_n]\,(t_0) \begin{pmatrix} I \\ A \\ \vdots \\ A^{n-1} \end{pmatrix}$$

and, by (3.5), the matrix exponential function is given by

$$e_A(t, t_0; \mathbb{T}) = [y_1(t)\,,\, y_2(t)\,,\, \ldots\,,\, y_n(t)]\, V^{-1}[y_1, \ldots, y_n](t_0) \begin{pmatrix} I \\ A \\ \vdots \\ A^{n-1} \end{pmatrix}. \tag{3.6}$$

We apply the Harris Algorithm to the following example: Consider the vector equation

$$x^{\Delta} = Ax, \quad \text{where } A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}. \tag{3.7}$$

The eigenvalues of A are $\lambda_1 = \lambda_2 = 3$ and thus A is regressive on any time scale. Moreover, A has only one linearly independent eigenvector. Then (3.4) reads $y^{\Delta\Delta} - 6y^{\Delta} + 9y = 0$ and we choose the solutions [3, Thm. 3.34] $y_1(t) = e_3(t, t_0; \mathbb{T})$ and $y_2(t) = e_3(t, t_0; \mathbb{T}) \int_{t_0}^{t} \frac{1}{1+3\mu(s)} \Delta s$ for some $t_0 \in \mathbb{T}$. Taking the delta derivative implies

$$V[y_1, y_2](t_0) = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

and therefore

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} I \\ A \end{pmatrix} = \begin{pmatrix} I \\ -3I + A \end{pmatrix}$$

and

$$e_A(t, t_0; \mathbb{T}) = e_3(t, t_0; \mathbb{T}) I + e_3(t, t_0; \mathbb{T}) \int_{t_0}^{t} \frac{1}{1 + 3\mu(s)} \Delta s \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Remark 10: Using techniques similar to the ones used in section 2.2, one can for some time scales derive the asymptotic behavior of the term $\int_{t_0}^{t} \frac{1}{1+\lambda\mu(s)} \Delta s$. Here we require, of course, that $1 + \lambda\mu(t) \neq 0$ for all $t \in \mathbb{T}$. For example, one can find for the time scales in Table 1 "between" $\mathbb{R}$ and N_0, i.e., $\mathbb{T} = \ln \mathbb{N}$ and $\mathbb{T} = \mathbb{N}_0^p$ with $0 < p \leq 2/3$, that $\int_{t_0}^{t} \frac{1}{1+\lambda\mu(s)} \Delta s = (t - t_0)[1 + \mathrm{o}(1)]$ as $t \to \infty$. The same asymptotic behavior can be found for the implicitly defined time scales with $\mu(t) = \frac{\kappa}{t^{\rho-1}} + \mathrm{O}\left(\frac{1}{t^{\gamma-1}}\right)$ with $\gamma > \rho > 2$ and $\kappa > 0$. Thus one can determine in these cases the asymptotic behavior of the matrix exponential function in two dimensions in the case of equal eigenvalues. For higher dimensional cases and other time scales, similar results may be obtained.

Remark 11: The Putzer algorithm is a special case of the Harris algorithm. That is, the solutions $r_j(t)$ $(1 \leq j \leq n)$ of the IVP (3.2) all satisfy (3.4) and $W(r_1, \ldots r_n)(t_0) = 1 \neq 0$. To establish our claim, observe that $(D - \lambda_j) r_j = r_{j-1}$ for $2 \leq j \leq n$ and $(D - \lambda_1) r_1 = 0$. Thus $\{r_1, \ldots, r_n\}$ are indeed solutions of (3.4). Invoking the dynamic equation for $r_j(t)$ given in (3.2), an inductive argument shows for $0 \leq k \leq j - 1$ that $D^k r_j = \sum_{i=0}^{k-1} a_{ji}(k) r_{j-i} + r_{j-k}$ with some coefficients $a_{ji}(k)$. We remark that the coefficient of r_{j-k} is equal to one. Utilizing the IVP given in (3.2), one finds that $(D^k r_j)(t_0) = r_{j-k}(t_0)$ for $0 \leq k \leq j-1$. It follows that $V(r_1, \ldots r_n)(t_0)$ is a lower triangular matrix with diagonal elements $V_{ii}(t_0) = 1$ for all $1 \leq i \leq n$ and the Wronski determinant $W(r_1, \ldots, r_n)(t_0) = 1 \neq 0$. Since the matrices F_i in the Harris algorithm are unique, they must coincide with the matrices P_i from the Putzer algorithm $(1 \leq i \leq n)$. This establishes our claim.

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