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POSITIVE EIGENVALUES FOR MATRIX DIFFERENCE EQUATIONS

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ABSTRACT. We consider a self-adjoint matrix difference eigenvalue problem with Dirichlet or anti-periodic boundary conditions. By reformulation of the problem as a Stieltjes Sturm–Liouville equation and then applying recent inequalities developed by Brown, Clark, and Hinton, conditions are given which imply all eigenvalues are positive of the difference equation problem. Examples are given which illustrate how these conditions allow cancellation of the positive and negative parts of the coefficients in the difference equation to preserve the positivity of the eigenvalues.

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1. INTRODUCTION

For a sequence of real r-vectors $y_k \in \mathbb{R}^r = \mathbb{R}^{r\times 1}$, $k = 0, 1, \ldots$, we consider the second order difference operator given by

$$L[y_k] = -\Delta^2 y_{k-1} + q_k y_k, \qquad k = 1, 2, \ldots \tag{1.1}$$

for which $q_k \in \mathbb{R}^{r\times r}$ and $q_k^t = q_k$ where q_k^t denotes the transpose of the real $r \times r$ matrix q_k, where $\Delta y_{k-1} = y_k - y_{k-1}$, and hence where $\Delta^2 y_{k-1} = y_{k+1} - 2y_k + y_{k-1}$. The eigenvalue problems studied are either of Dirichlet type as given by

$$L[y_k] = \lambda w_k y_k, \quad k = 1, \ldots, n, \quad y_0 = y_{n+1} = 0 \tag{1.2}$$

or of anti-periodic type, as given by

$$L[y_k] = \lambda w_k y_k, \quad k = 1, \ldots, n, \quad y_0 + y_{n+1} = 0, \quad \Delta y_0 + \Delta y_n = 0, \tag{1.3}$$

where $w_k \in \mathbb{R}^{r\times r}$, and $w_k > 0$, with the later condition denoting positive definiteness for the real $r \times r$ matrix w_k.

Both (1.2) and (1.3) have equivalent matrix formulations, are self-adjoint, and possess a finite number of eigenvalues. In this paper we give conditions sufficient to guarantee that the least eigenvalues of (1.2) and (1.3) are positive. Positivity of eigenvalues for boundary value problems like those described in (1.2) and (1.3)

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can be related to the stability and nonoscillation of solutions for the Sturm–Liouville eigenvalue equation in the discrete and continuous case as can be seen in Clark and Hinton [4, 5, 6].

When all $q_k = 0$ the eigenvalues for (1.2) and for (1.3) are positive; this case, of course, is covered by our theorems. However, it should be noted that, when elements of q_k vary in sign, our criteria allow for cancellation of the positive and negative parts of q_k to preserve positivity of eigenvalues.

Our conditions are stated in terms of a parameter p, for $1 \le p \le 2$. An analog of the $p = 1$ case may be found in Clark and Hinton [4, 6]. However, the discrete inequalities employed there do not appear to have extensions to the situations where $1 < p \le 2$. To obtain our conditions for positivity of eigenvalues, we employ integral inequalities for appropriate classes of continuous, real-valued functions. These inequalities are stated in Section 2 and were recently used by Brown, Clark and Hinton [3] to study positivity of eigenvalues for vector-matrix differential equations as well as oscillation for Stieltjes Sturm–Liouville equations. Through formulations of (1.2) and (1.3) as Stieltjes Sturm–Liouville eigenvalue problems, we use the integral inequalities of Section 2 to obtain conditions for positivity of eigenvalues in the analogous discrete case.

We associate with the sequence y_k, $k = 0, 1, 2, \dots$, the continuous function

$$u(x) = \begin{cases} y_0, & x < 0, \\ y_k + (x-k)\Delta y_k, & k \le x < k+1. \end{cases} \tag{1.4}$$

For the interval $[0, n+1)$, we define the matrix-valued step functions Q and W by

$$Q(x) = \begin{cases} 0, & x < 1, \\ \sum_{k=1}^{\lfloor x \rfloor} q_k, & 1 \le x < n+1, \end{cases} \quad W(x) = \begin{cases} 0, & x < 1, \\ \sum_{k=1}^{\lfloor x \rfloor} w_k, & 1 \le x < n+1, \end{cases} \tag{1.5a}$$

where $\lfloor x \rfloor$ is the greatest integer function. Since the values of q_{n+1} and w_{n+1} play no role in either (1.2) or (1.3) we define Q and W by continuity at $x = n+1$, e.g.

$$Q(n+1) = Q(n), \qquad W(n+1) = W(n). \tag{1.5b}$$

Finding a solution of $L[y_k] = \lambda w_k y_k$ is then equivalent to finding an absolutely continuous function u defined on $[0, n+1]$ such that in $(0, n+1)$, except at $x = 1, \dots, n$,

$$\Lambda[u] = -u'(x) + u'(0) + \int_0^x dQ(s)u(s) = \lambda \int_0^x dW(s)u(s), \tag{1.6}$$

where in (1.6) the integrals are Riemann Stieltjes integrals and where at the points $0, 1, \dots, n+1$, (1.6) holds in the sense of left and right limits. Thus problem (1.2) is equivalent to finding an absolutely continuous function u defined on $[0, n+1]$ such that

$$\Lambda[u] = \lambda \int_0^x dW(s)u(s), \qquad u(0) = u(n+1) = 0, \tag{1.7}$$

with equality holding at $x = 1, \ldots, n$ in terms of left and right limits. Problem (1.3) is equivalent to finding an absolutely continuous function u similarly defined on $[0, n+1]$ such that

$$\Lambda[u] = \lambda \int_0^x dW(s)u(s), \qquad u(0) + u(n+1) = 0, \quad u'(0) + u'(n+1) = 0. \tag{1.8}$$

A general theory for (1.7) may be found in Reid [8] where the emphasis is on oscillation theory. In particular, a unique solution u exists with $u(0)$, $u'(0)$ given under the condition that Q, W are of bounded variation. For the scalar, $r = 1$, case, a rather complete spectral theory for (1.7) is developed by Mingarelli [7].

2. INEQUALITIES

All functions in this paper will be either real, real vector-valued, or real matrix-valued. We enforce the following notational conventions: q^t denotes the transpose of the matrix q. If $1 \le r \le \infty$, then r' is the conjugate index of r, i.e., $\frac{1}{r} + \frac{1}{r'} = 1$. $\mathbb{R}^n$ denotes n-dimensional Euclidean space with norm $|\cdot|$.

We now define classes of functions used throughout the paper.

$$\begin{aligned}
AC^r[a,b] &= \{f : [a,b] \to \mathbb{R}^r \mid f \text{ is absolutely continuous on } [a,b]\} \\
\mathcal{L}_p^r[a,b] &= \{f : [a,b] \to \mathbb{R}^r \mid f \text{ is Lebesgue measurable, } \textstyle\int_a^b |f|^p \, dx < \infty\} \\
\mathcal{D}_d^r[a,b] &= \{f \in AC^r[a,b] \mid f(a) = f(b) = 0, \ f' \in \mathcal{L}_2^r[a,b]\} \\
\mathcal{D}_{ap}^r[a,b] &= \{f \in AC^r[a,b] \mid f(a) + f(b) = 0, \ f' \in \mathcal{L}_2^r[a,b]\} \, .
\end{aligned}$$

Note that $\mathcal{D}_d^r[a,b] \subset \mathcal{D}_{ap}^r[a,b]$. Note also that when $r = 1$, the superscript r will be omitted; similarly for $p = 1$ in $\mathcal{L}_p^r[a,b]$. We also let the inner product for the Hilbert space $\mathcal{L}_2^r[a,b]$ be denoted by $\langle \cdot, \cdot \rangle$ with the norm denoted by $\|\cdot\|$.

Next, we collect some inequalities which serve as the foundation of our later analysis. The first is a result of Boyd.

Theorem 2.1 (Boyd, [1])**.** *Suppose $f \in AC[a,b]$, $f(a) = 0$ or $f(b) = 0$, $1 \le p \le 2$, and $f' \in \mathcal{L}_2[a,b]$. Then*

$$\int_a^b |ff'|^p \, dx \le K(p)(b-a) \left(\int_a^b |f'|^2 \, dx \right)^p$$

where

$$K(p) = \begin{cases} 1/2, & p = 1 \\ 4/\pi^2, & p = 2 \\ \big((2-p)/2\big) I^{-p}, & 1 < p < 2, \end{cases} \tag{2.1}$$

and

$$I = \int_0^1 \frac{\big(1 + (p-1)t\big) t^{\frac{1}{p}-1}}{\Big(1 + \big(2(p-1)/(2-p)\big)t\Big)^2}\, dt. \tag{2.2}$$

Moreover, the constant $K(p)$ *is sharp.*

Boyd proved this theorem on $[a, b] = [0, 1]$. The general case above follows by a simple scaling. The function $K(p)$ is continuous on $1 \le p \le 2$. This can be shown by the change of variable in (2.2), $x = t\big/(2-p)$, $1 < p < 2$, which yields that

$$K(p) = \frac{1}{2}\left(\int_0^{1/(2-p)} \frac{\big(1 + (p-1)(2-p)x\big) x^{\frac{1}{p}-1}}{\big(1 + 2(p-1)x\big)^2}\, dx\right)^{-p}.$$

Clearly, $K(p) \to 1/2$ as $p \to 1^+$ and as $p \to 2^-$, $K(p)$ approaches

$$\frac{1}{2}\left(\int_0^{\infty} \frac{1}{\sqrt{x}(1+2x)^2}\, dx\right)^{-2} = \frac{4}{\pi^2}.$$

The next three results were proven by Brown, Clark and Hinton in [3] where the constant $K(p)$ is defined in Theorem 2.1. The first of the following results, Theorem 2.2, represents an extension of Theorem 2.1 to the function class $\mathcal{D}_{ap}[a, b]$. The second and third results are two-function extensions of Theorem 2.2.

Theorem 2.2. *Suppose* $f \in \mathcal{D}_{ap}[a, b]$, *and* $1 \le p \le 2$. *Then*

$$\int_a^b |ff'|^p\, dx \le K(p)\frac{b-a}{2^p}\left(\int_a^b |f'|^2\, dx\right)^p. \tag{2.3}$$

Moreover, the constant in (2.3) *is sharp.*

Theorem 2.3. *Suppose* $f, g \in \mathcal{D}_d[a, b]$ *and* $1 \le p \le 2$. *Then,*

$$\int_a^b \big(|f'g| + |g'f|\big)^p\, dx \le K(p)\frac{b-a}{2^p}\left(\int_a^b |f'|^2 + |g'|^2\, dx\right)^p.$$

Theorem 2.4. *Suppose* $f, g \in \mathcal{D}_{ap}[a, b]$ *and* $1 \le p \le 2$. *Then,*

$$\int_a^b |f'g + g'f|^p\, dx \le K(p)\frac{b-a}{2^p}\left(\int_a^b |f'|^2 + |g'|^2\, dx\right)^p.$$

Before stating the last inequality, we consider the 1-dimensional Sobolev inequality, c.f. Talenti [9].

Theorem 2.5. *Suppose* $1 \le \nu \le \infty$, $1 < \mu < \infty$, $f \in AC[a,b]$ *with* $f(a) = f(b) = 0$ *and* $f' \in \mathcal{L}_\mu[a,b]$. *Then*

$$\left(\int_a^b |f|^\nu \, dx\right)^{1/\nu} \le K_s(\nu,\mu)(b-a)^{1+\frac{1}{\nu}-\frac{1}{\mu}} \left(\int_a^b |f'|^\mu \, dx\right)^{1/\mu} \tag{2.4}$$

where for $\nu \ne \infty$,

$$K_s(\nu,\mu) = \frac{\nu\left(1+\frac{\mu'}{\nu}\right)^{1/\mu} \Gamma\left(\frac{1}{\nu}+\frac{1}{\mu'}\right)}{2\left(1+\frac{\nu}{\mu'}\right)^{1/\nu} \Gamma\left(\frac{1}{\nu}\right)\Gamma\left(\frac{1}{\mu'}\right)}, \tag{2.5}$$

and where Γ *represents the gamma function and* μ' *is the conjugate index of* μ. *Furthermore,* $K_s(\infty,\mu) = 1/2$ *and* $K(2,2) = 1/\pi$, *and the constant* $K_s(\nu,\mu)$ *is sharp in* (2.4).

Remark 2.6. An examination of the proof of Theorem 2.2 given in [3] shows that (2.4) also holds when the boundary conditions $f(a) = f(b) = 0$ are replaced with the antiperiodic boundary condition $f(a) + f(b) = 0$.

Our last inequality represents an analog of Theorem 2.3 for the function class $\mathcal{D}_{ap}[a,b]$; its proof is given in [3].

Theorem 2.7. *Suppose that* $1 \le p \le 2$, *that* $\nu = 2p/(2-p)$, *and that* $f, g \in \mathcal{D}_{ap}[a,b]$. *Then,*

$$\begin{aligned} &\int_a^b \left(|f'g| + |g'f|\right)^p \, dx \\ &\qquad \le 2^p K_s(\nu,2)^p (b-a) \left(\int_a^b |f'|^2 \, dx\right)^{p/2} \left(\int_a^b |g'|^2 \, dx\right)^{p/2}. \end{aligned} \tag{2.6}$$

3. POSITIVITY OF EIGENVALUES

If f is in $\mathcal{L}_v[a,b]$ and $1 \le v \le \infty$, we define

$$N(f,v) = \inf_{-\infty<\gamma<\infty} \left(\int_a^b |f(s)+\gamma|^v \, ds\right)^{1/v} \tag{3.1}$$

where for $v = \infty$ the supremum norm is meant in (3.1). In general, while it is difficult to compute (3.1) upper bounds for $N(f,v)$ may be obtained by choice of γ. There are two exceptions to the general difficulty of computing $N(f,v)$ in (3.1). When $v = \infty$, it is proved in [2] that with $f(x) = \int_a^x q(s)\,ds$, and $q \in \mathcal{L}[a,b]$,

$$N(f,\infty) = \inf_{-\infty<\gamma<\infty} \left(\max_{a\le x\le b} |f(x)+\gamma|\right) = \frac{M-m}{2} = \frac{1}{2}\left|\int_{t_1}^{t_2} q(s)\,ds\right| \tag{3.2}$$

where m, M, t_1, and t_2 are such that

$$m = \min_{a\le x\le b} f(x) \equiv \int_a^{t_1} q(s)\,ds\,, \qquad M = \max_{a\le x\le b} f(x) \equiv \int_a^{t_2} q(s)\,ds\,,$$

and where the optimal choice for γ is $-(M+m)/2$. For $v=2$, by setting $\gamma = -\frac{1}{(b-a)}\int_a^b\int_a^x q(s)\,dsdx$, we obtain that

$$N(f,2)^2 = \int_a^b \left(\int_a^x q(s)\,ds\right)^2 dx - \frac{1}{b-a}\left(\int_a^b\int_a^x q(s)\,dsdx\right)^2. \tag{3.3}$$

For our applications, f will be a right continuous step function on $[0,n+1)$ with jumps at $1,\ldots,n$ and with continuity at $n+1$. For this f,

$$\left(\int_0^{n+1} |f(s)+\gamma|^v ds\right)^{1/v} = \left(\sum_{k=0}^n |f(k)+\gamma|^v ds\right)^{1/v}.$$

In particular, (3.2) and (3.3) become, respectively,

$$N(f,\infty) = \frac{M-m}{2}, \qquad m = \min_{0\le k\le n} f(k), \quad M = \max_{0\le k\le n} f(k),$$

and

$$N(f,2)^2 = \sum_{k=0}^n f(k)^2 - \frac{1}{n+1}\left(\sum_{k=0}^n f(k)\right)^2.$$

For the statement of our results, we need the following auxiliary functions and matrices: With $\Omega\in\mathbb{R}^{r\times r}$, $\Omega^t=\Omega$, and with $Q(x)$ as defined in (1.5) define

$$R(x) = Q(x)+Q(x)^t+\Omega = 2Q(x)+\Omega, \qquad 0\le x\le n+1. \tag{3.4}$$

Define the real, $r\times r$ matrices β and χ by

$$\beta_{ij} = N(2Q_{ij},p'), \qquad \chi_{ij} = \begin{cases} \beta_{ii}\big(K(p)\big)^{1/p}/2, & i=j \\ \beta_{ij}K_s(v,2), & i\ne j, \end{cases} \tag{3.5}$$

where $1\le p\le 2$, $v=2p/(2-p)$, and p' is the conjugate index for p, and where $K(p)$ is defined in (2.1) and $K_s(\nu,2)$ is defined in (2.5).

We let $|v|_\infty = \max_{1\le i\le r}|v_i|$, for $v\in\mathbb{R}^r$, and let $\|\beta\|_\infty$ denote the maximum row-sum norm for $\beta\in\mathbb{R}^{r\times r}$:

$$\|\beta\|_\infty = \max_{1\le i\le r}\sum_{j=1}^r \beta_{ij} = \sup_{|v|_\infty=1}|\beta v|_\infty.$$

Theorem 3.1. *Let $w_k>0$, q_k, $k=1,2,\ldots$ be symmetric, real, $r\times r$ matrices, and let $Q(x)$ be the symmetric, real, $r\times r$ matrix-valued step function defined in* (1.5) *for the interval* $[0,n+1]$*. Then, the eigenvalues of* (1.3) *(c.f.* (1.8)*) when $Q(n)\ge 0$, and the eigenvalues of* (1.2) *(c.f.* (1.7)*) when $Q(n)$ is arbitrary, are positive when either*

$$\|\beta\|_\infty^p K(p)\frac{n+1}{2^p} < 1 \tag{3.6}$$

or

$$\delta^p(n+1) < 1 \tag{3.7}$$

hold for some p, $1 \le p \le 2$, where β and χ are the matrices defined in terms of $Q(x)$ in (3.5)*, and where δ is the largest eigenvalue of the matrix χ.*

Proof. If y_k, $k = 0, 1, \ldots$ is a solution of either (1.2) or (1.3), then

$$\begin{aligned}
\lambda \sum_{k=1}^{n} y_k^t w_k y_k &= \sum_{k=1}^{n} \big(- y_k^t \Delta^2 y_{k-1} + y_k^t q_k y_k \big) \\
&= -y_{n+1}^t \Delta y_n + y_1^t \Delta y_0 + \sum_{k=1}^{n} \big((\Delta y_k)^t \Delta y_k + y_k^t q_k y_k \big) \\
&= \sum_{k=0}^{n} (\Delta y_k)^t \Delta y_k + \sum_{k=1}^{n} y_k^t q_k y_k,
\end{aligned}$$

where we have used the boundary conditions in (1.2) and (1.3) to obtain

$$-y_{n+1}^t \Delta y_n + y_1^t \Delta y_0 = (\Delta y_0)^t \Delta y_0.$$

Given the definition of u in (1.4), and that of W and Q given in (1.5), and recalling that W and Q are left continuous at $n+1$, note that

$$\sum_{k=1}^{n} y_k^t w_k y_k = \int_0^{n+1} u(s)^t dW(s) u(s), \quad \sum_{k=1}^{n} y_k^t q_k y_k = \int_0^{n+1} u(s)^t dQ(s) u(s)$$

and that

$$\sum_{k=0}^{n} (\Delta y_k)^t \Delta y_k = \|u'\|^2,$$

where

$$\|u'\|^2 = \int_0^{n+1} u'(s)^t u'(s) ds.$$

Hence, the positivity of the eigenvalues of (1.2) and (1.3) (c.f. (1.7) resp. (1.8)) will follow from the positivity of the functional $J[u]$ defined by

$$\begin{aligned}
J[u] &= \int_0^{n+1} u'(s)^t u'(s) ds + \int_0^{n+1} u(s)^t dQ(s) u(s), \\
&= \sum_{k=0}^{n} (\Delta y_k)^t \Delta y_k + \sum_{k=1}^{n} y_k^t q_k y_k,
\end{aligned} \tag{3.8}$$

where J is defined on $\mathcal{D}_d[0, n+1]$ for (1.2) and defined on $\mathcal{D}_{ap}[0, n+1]$ for (1.3).

Toward this end, we note that integration by parts for an absolutely continuous function $u \in \mathcal{D}_{ap}^r[0, n+1]$ gives

$$\int_0^{n+1} u(s)^t \, dQ(s) u(s) = u(0)^t Q(n) u(0) - \int_0^{n+1} u'(s)^t R(s) u(s) \, ds,$$

where the boundary conditions in (1.2) and (1.3) and the definition of Q in (1.5) have been used to obtain

$$\int_0^{n+1} \big(u'(s)^t \Omega u(s) + u(s)^t \Omega u'(s) \big) ds = u(s)^t \Omega u(s) \Big|_0^{n+1} = 0,$$

and

$$u(s)^t Q(s) u(s)\Big|_0^{n+1} = u(0)^t Q(n) u(0).$$

Thus, for $u \in \mathcal{D}^r_{ap}[0, n+1]$ with $Q(n) \geq 0$, or for $u \in \mathcal{D}^r_d[0, n+1]$ with $Q(n)$ arbitrary, (3.8) becomes

$$J[u] = \|u'\|^2 + u(0)^t Q(n) u(0) - \int_0^{n+1} u'(s)^t R(s) u(s)\, ds \tag{3.9}$$

$$\geq \|u'\|^2 - \int_0^{n+1} u'(s)^t R(s) u(s)\, ds. \tag{3.10}$$

Next, we obtain two estimates using the inequalities of Section 2: With $u \in \mathcal{D}^r_{ap}[0, n+1]$, and with R as it is defined in (3.4)

$$\begin{aligned}
&\Big| \int_0^{n+1} u'(s)^t R(s) u(s)\, ds \Big| \\
&\quad\leq \sum_{i=1}^n \Big(\int_0^{n+1} |u_i' u_i|^p\, ds \Big)^{1/p} \Big(\int_0^{n+1} |R_{ii}(s)|^{p'}\, ds \Big)^{1/p'} \\
&\qquad + \sum_{i>j} \Big(\int_0^{n+1} |u_i' u_j + u_i u_j'|^p\, ds \Big)^{1/p} \Big(\int_0^{n+1} |R_{ij}(s)|^{p'}\, ds \Big)^{1/p'} \\
&\quad\leq \sum_{i=1}^n \Big(\int_0^{n+1} |u_i' u_i|^p\, ds \Big)^{1/p} \beta_{ii} \\
&\qquad + \sum_{i>j} \Big(\int_0^{n+1} |u_i' u_j + u_i u_j'|^p\, ds \Big)^{1/p} \beta_{ij}.
\end{aligned} \tag{3.11}$$

Applying Theorems 2.2 and 2.4 to the right side of (3.11), we obtain

$$\begin{aligned}
&\Big| \int_0^{n+1} u'(s)^t R(s) u(s)\, ds \Big| \\
&\quad\leq \sum_{i=1}^n \beta_{ii} \frac{1}{2} \big(K(p)(n+1) \big)^{1/p} \int_0^{n+1} |u_i'|^2\, ds \\
&\qquad + \sum_{i>j} \beta_{ij} \frac{1}{2} \big(K(p)(n+1) \big)^{1/p} \int_0^{n+1} \big(|u_i'|^2 + |u_j'|^2 \big)\, ds \\
&\quad= \frac{1}{2} \big(K(p)(n+1) \big)^{1/p} \sum_{i=1}^n \sum_{j=1}^n \beta_{ij} \int_0^{n+1} |u_i'|^2\, ds \\
&\quad\leq \frac{1}{2} \big(K(p)(n+1) \big)^{1/p} \|\beta\|_\infty \|u'\|^2.
\end{aligned} \tag{3.12}$$

On the other hand, if we apply Theorems 2.2 and 2.7 to the right side of (3.11), we obtain

$$\begin{aligned}
&\Big|\int_0^{n+1} u'(s)^t R(s)u(s)\,ds\Big| \\
&\qquad \le \sum_{i=1}^{n} \beta_{ii}\frac{1}{2}\big(K(p)(n+1)\big)^{1/p}\int_0^{n+1}|u_i'|^2\,ds \\
&\qquad\qquad + \sum_{i>j} 2\beta_{ij}K_s(v,2)(n+1)^{1/p}\left(\int_0^{n+1}|u_i'|^2 ds\int_0^{n+1}|u_j'|^2 ds\right)^{1/2} \\
&\qquad \le (n+1)^{1/p}\delta\|u'\|^2\ , \qquad (3.13)
\end{aligned}$$

where δ is the largest eigenvalue of the matrix χ which is defined in (3.5).

Thus, for $u \in \mathcal{D}^r_{ap}[0,n+1]$ with $Q(n) \ge 0$, or for $u \in \mathcal{D}^r_d[0,n+1]$ with $Q(n)$ arbitrary, (3.10), (3.12), and (3.13) imply that $J[u] > 0$ when either (3.6) or (3.7) is satisfied. As noted earlier, positivity of $J[u]$ implies positivity of the eigenvalues for the boundary value problems given in (1.2) and (1.3) or equivalently (1.7) and respectively (1.8). □

A continuation of the previous result follows.

Theorem 3.2. *Let $w_k > 0$, q_k, $k = 1, 2, \ldots$ be symmetric, real, $r \times r$ matrices, and let $Q(x)$ be the symmetric, real, $r\times r$ matrix-valued step function defined in* (1.5) *for the interval $[0, n+1]$. Let ω be the smallest eigenvalue of $Q(n)$. Then, when $\omega < 0$, the eigenvalues of* (1.3) *(c.f.* (1.8)*) are positive when either*

$$\frac{1}{2}\big(K(p)(n+1)\big)^{1/p}\|\beta\|_\infty + \frac{1}{4}(n+1)|\omega| < 1 \qquad (3.14)$$

or

$$\delta(n+1)^{1/p} + \frac{1}{4}(n+1)|\omega| < 1 \qquad (3.15)$$

hold for some p, $1 \le p \le 2$, where β and χ are the matrices defined in terms of $Q(x)$ in (3.5)*, and where δ is the largest eigenvalue of the matrix χ.*

Proof. For $u \in \mathcal{D}^r_{ap}[0,n+1]$ with $\omega < 0$, using (3.9) we obtain

$$J[u] \ge \omega|u(0)|^2 + \|u'\|^2 - \int_0^{n+1} u'(s)^t R(s)u(s)\,ds. \qquad (3.16)$$

Since $-2u(0) = u(n+1) - u(0) = \int_0^{n+1} u'(s)ds$, it follows by the Cauchy-Schwarz inequality that

$$|u(0)|^2 \le \frac{n+1}{4}\|u'\|^2.$$

It then follows from (3.11) and (3.12) that the right side of (3.16) is positive, i.e. that $J[u] > 0$, when (3.14) or (3.15) hold for $1 \le p \le 2$, thereby showing that under these circumstances the eigenvalues of (1.3), (c.f. (1.8)), are positive. □

4. EXAMPLES

We conclude with examples illustrating the fact that different tests for the positivity of eigenvalues for (1.2) and (1.3) may result from different values of $1 \le p \le 2$. The first example gives a scalar difference expression for which not all of the values of the parameter $1 \le p \le 2$ reveal the fact that the eigenvalues for (1.2) and (1.3) are positive.

Example 4.1. Suppose $\epsilon > 0$ and $r = 1$ in (1.1). For $r = 1$, (3.6) and (3.7) are the same. Set $q_2 = -\epsilon$ and $q_k = 0$ for $k \neq 0$. Then $Q(1) = 0$ and $Q(k) = -\epsilon$ for $k \ge 2$ and it follows that $M = \max_{1\le k\le n} Q(k) = 0$ and $m = \min_{1\le k\le n} Q(k) = -\epsilon$. When $p = 2$, (3.7) holds if

$$\begin{aligned} 1 > \delta^2(n+1) &= \frac{4(n+1)}{\pi^2}\left(\sum_{k=0}^{n} Q(k)^2 - \frac{1}{n+1}\left(\sum_{k=0}^{n} Q(k)\right)^2\right) \\ &= \frac{4(n+1)}{\pi^2}\left((n-1)\epsilon^2 - \frac{(n-1)^2\epsilon^2}{n+1}\right) \\ &= \frac{8(n-1)\epsilon^2}{\pi^2}. \end{aligned}$$

Note that (3.7) holds for $p = 1$ if $\delta(n+1) = \frac{\epsilon(n+1)}{4} < 1$. By choosing ϵ so that $\frac{\epsilon(n+1)}{4} > 1$, e.g., $\epsilon = \frac{5}{n+1}$ and choosing n so that

$$1 > \frac{8(n-1)\epsilon^2}{\pi^2} = \frac{200(n-1)}{\pi^2(n+1)^2},$$

we have an example where (3.7) fails for $p = 1$ and holds for $p = 2$.

In the next example a vector-matrix difference expression is given where our tests obtained for positivity of eigenvalues for (1.2) and (1.3) depend not only on p, but on whether (3.6) or (3.7) is used.

Example 4.2. Let $\epsilon > 0$ and let B be the $r \times r$ matrix with $B_{ij} = 0$ for $i = j$ and 1 otherwise. Since $B + I$ has eigenvalues r and 0 (multiplicity $r - 1$), B has eigenvalues $r - 1$ and -1 (multiplicity $r - 1$). Let $q_k = (-1)^k \epsilon B$. Then $(Q(k))_{ii} = 0$ and for $i \neq j$, $(Q(k))_{ij} = -\epsilon$ for k odd and $(Q(k))_{ij} = 0$ for k even. We only consider the case n is odd; the case n is even is similar. For $p = 1$, $\beta_{ii} = 0$, and for $i \neq j$,

$$\beta_{ij} = \max_{1\le k\le n} (Q(k))_{ij} - \min_{1\le k\le n} (Q(k))_{ij} = \epsilon,$$

and for $p = 2$, $\beta_{ii} = 0$, and for $i \neq j$,

$$\begin{aligned}\beta_{ij} &= 2\left(\sum_{k=0}^{n}(Q(k))_{ij}^2 - \frac{1}{n+1}\left(\sum_{k=0}^{n}(Q(k))_{ij}\right)^2\right)^{1/2}\\ &= 2\left(\frac{(n+1)\epsilon^2}{2} - \frac{1}{n+1}\left(\frac{(n+1)\epsilon}{2}\right)^2\right)^{1/2}\\ &= (n+1)^{1/2}\epsilon.\end{aligned}$$

Hence for $p = 1$, $\|\beta\|_\infty = (r-1)\epsilon, \chi = K_s(2,2)\beta = \beta/\pi$, so that $\delta = (r-1)\epsilon/\pi$. The equations (3.6) and (3.7) become respectively,

$$1 > \|\beta\|_\infty \frac{K(1)(n+1)}{2} = \frac{(r-1)(n+1)\epsilon}{4},$$

and

$$1 > \delta(n+1) = \frac{(r-1)(n+1)\epsilon}{\pi}.$$

For $p = 2$, $\|\beta\|_\infty = (r-1)(n+1)^{1/2}\epsilon, \chi = K_s(\infty,2)\beta$, so that $\delta = \|\beta\|_\infty/2$. The equations (3.6) and (3.7) become respectively,

$$1 > \frac{\|\beta\|_\infty}{2}\big(K(2)(n+1)\big)^{1/2} = \frac{(r-1)(n+1)\epsilon}{\pi},$$

and

$$1 > \delta(n+1)^{1/2} = \frac{(r-1)(n+1)\epsilon}{2}.$$

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