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A CONTINUATION OF THE DISCUSSION ON CROSS SYMMETRY OF SOLUTIONS

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ABSTRACT. In this paper we continue to explore cross symmetry properties of the solutions of second-order nonlinear boundary value problems on time scales. Dynamic equations under delta and nabla differentiations are considered. It is proven that, by introducing a proper companion problem, the solution of a dynamic equation is cross symmetric to the solution of the companion problem. Proper jump functions on time scales are utilized. Computational examples are given to further illustrate our conclusions.

AMS (MOS) Subject Classification. 34B10, 39A10.

1. INTRODUCTION

Beginning with the work of Hilger [5], there has been considerable activity in the study of dynamic equations on time scales. We refer the reader to the recently published monograph [3] and the extensive bibliography produced in [3]. Initially, Hilger's motivation to define time scales and Δ differentiability on time scales was to unify the traditional calculus on intervals and the calculus of finite differences. The process to unify the discrete and the continuous has been swift and remarkably successful as recorded in [3].

A second motivation to continue the development of analysis on time scales is to extend the traditional calculus and the calculus of finite differences. It is extension that motivates our study of cross symmetry properties of solutions of boundary value problems (BVPs) on time scales. When one thinks of the calculus of finite differences, one often thinks of time scales with constant (and positive) graininess. Adaptive methods are prevalent in computational methods and hence are not addressed in the case of constant graininess; adaptive methods can be addressed in the case of time scales.

In this paper, we continue a study we began in [4]. We initially intended to study the symmetry properties of solutions of dynamic equations on time scales. We actually showed that symmetry is not a property to be expected. Instead, we showed

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that a given BVP has a companion BVP and that respective solutions of the BVP and the companion BVP are cross symmetric (a term we will define again, below).

This observation has meaning in computation when applying finite difference methods to BVPs for ordinary differential equations. In particular, if the solution of the BVP for the differential equation is symmetric, a numerical solution, obtained through adaptive finite difference methods, is not. The numerical error bound may be well within an acceptable tolerance and yet, the numerical solution has lost a fundamental shape feature, symmetry. Although we did not point it out in [4], both the BVP and the companion BVP represent adaptive finite difference methods applied to the same BVP for a differential equation. Hence the solution of the BVP and the solution of the companion BVP represent numerical estimates for the solution of the BVP for the differential equation. Thus, the average of the two time scales solutions represents a numerical solution approximating the solution of the BVP for the differential equation. Finally, the average of the two time scales solutions maintains the fundamental shape feature, symmetry.

In this paper we shall develop this idea in a broader framework. Symmetry will be a special case. We point out that none of the concepts in the paper are difficult and yet, we believe they are new. The calculus on time scales has provided the tools to both bring the concept of cross symmetry to light and to analyze the concept of cross symmetry.

In Section 2, we provide some background material with respect to time scales. We do assume the reader is familiar with the calculus on time scales. In Section 3, we introduce the concepts of symmetry and anti-symmetry and argue that the average of two cross symmetric numerical solutions maintains a true solution's features with respect to symmetry and anti-symmetry. We also show the discussion applies to partial differential equations such as the reaction-diffusion equations. We close the paper in Section 4 with computational examples to illustrate our results.

2. TIME SCALE PRELIMINARIES

We assume the reader is familiar with standard notation in the calculus of time scales [3]. We shall use f^Δ to denote a forward or delta derivative and we shall use f^∇ to denote a backward or nabla derivative. Recall that if $\mathbb{T}$ denotes a time scale and for $t \in \mathbb{T}$ then

$$\sigma(t) = \inf\{s \in \mathbb{T}, s > t\}.$$

Recall: $\mathbb{T}^\kappa = \mathbb{T}$ if b is left-dense and $\mathbb{T}^\kappa = \mathbb{T} \setminus \{b\}$ if b is left-scattered. In [4] we define $\mathbb{T}_\kappa = \mathbb{T}$ if a is right-dense and $\mathbb{T}_\kappa = \mathbb{T} \setminus \{a\}$ if a is right-scattered.

The graininess at t is then defined by $\mu(t) = \sigma(t) - t$. Adaptive finite difference methods motivate the ideas in this paper; in particular, we are interested in time scales $\mathbb{T}$ with nonconstant graininess.

The chain rule is crucial to the development of the concept of cross symmetry. Ahlbrandt, Bohner, and Ridenhour [1] studied the change of variables in the calculus on time scales earlier in the cases where only delta derivatives are employed. Several useful relations are given. Based on their results and those by Bohner and Peterson [3], we obtained the following in [4].

Lemma 2.1. (Chain Rule) *Let $\nu: \mathbb{T} \to \mathbb{R}$ be monotone.*

(i): *Assume ν is strictly increasing and $\tilde{\mathbb{T}}: \nu(\mathbb{T})$ is a measure chain. Let $\omega: \tilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^{\Delta}(t)$ and $\omega^{\tilde{\Delta}}(\nu(t))$ exist for $t \in \mathbb{T}^k$, then*

$$(\omega \circ \nu)^{\Delta} = (\omega^{\tilde{\Delta}} \circ \nu)\nu^{\Delta}. \tag{2.1}$$

On the other hand, if $\nu^{\nabla}(t)$ and $\omega^{\tilde{\nabla}}(\nu(t))$ exist for $t \in \mathbb{T}_k$, then

$$(\omega \circ \nu)^{\nabla} = (\omega^{\tilde{\nabla}} \circ \nu)\nu^{\nabla}. \tag{2.2}$$

(ii): *Assume ν is strictly decreasing and $\tilde{\mathbb{T}}: -\nu(\mathbb{T})$ is a measure chain. Let $\omega: \tilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^{\Delta}(t)$ and $\omega^{\tilde{\nabla}}(\nu(t))$ exist for $t \in \mathbb{T}^k$, then*

$$(\omega \circ \nu)^{\Delta} = (\omega^{\tilde{\nabla}} \circ \nu)\nu^{\Delta}. \tag{2.3}$$

On the other hand, if $\nu^{\nabla}(t)$ and $\omega^{\tilde{\Delta}}(\nu(t))$ exist for $t \in \mathbb{T}_k$, then

$$(\omega \circ \nu)^{\nabla} = (\omega^{\tilde{\Delta}} \circ \nu)\nu^{\nabla}. \tag{2.4}$$

Once the chain rule is obtained, one can develop substitution rules in definite integrals. In the case of forward derivatives, the substitution formula is developed in [3]; the remaining three formulas are obtained in [4].

Theorem 2.2. (Substitution Rule) *Let $\nu: \mathbb{T} \to \mathbb{R}$ be monotone.*

(i): *Assume ν is strictly increasing and $\tilde{\mathbb{T}}: \nu(\mathbb{T})$ is a measure chain. If $f: \mathbb{T} \to \mathbb{R}$ is a continuous function and ν is differentiable with continuous derivative, then if $a, b \in \mathbb{T}$,*

$$\int_a^b f(t)\nu^{\Delta}(t)\Delta t = \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)\tilde{\Delta} s \tag{2.5}$$

or

$$\int_a^b f(t)\nu^{\nabla}(t)\nabla t = \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)\tilde{\nabla} s. \tag{2.6}$$

(ii): *Assume ν is strictly decreasing and $\tilde{\mathbb{T}}: -\nu(\mathbb{T})$ is a measure chain. If $f: \mathbb{T} \to \mathbb{R}$ is a continuous function and ν is differentiable with continuous derivative, then if $a, b \in \mathbb{T}$,*

$$\int_a^b f(t)\nu^\Delta(t)\Delta t = \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)\tilde{\nabla} s \tag{2.7}$$

or

$$\int_a^b f(t)\nu^\nabla(t)\nabla t = \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)\tilde{\Delta} s. \tag{2.8}$$

Throughout this paper we assume that the time scale $\mathbb{T}$ is bounded. Let

$$a = \min \mathbb{T}, \quad b = \max \mathbb{T}.$$

Moreover, we shall assume $\mathbb{T}$ is symmetric; that is

$$t \in \mathbb{T} \Longleftrightarrow a + b - t \in \mathbb{T}. \tag{2.9}$$

The following lemma was proved in [4].

Lemma 2.3. *Let $t \in \mathbb{T}^\kappa$ and let $u = a + b - t$. Then*

$$\sigma(t) = a + b - \rho(u).$$

On the other hand, if $t \in \mathbb{T}_\kappa$ then

$$\rho(t) = a + b - \sigma(u).$$

3. COMPANION BOUNDARY VALUE PROBLEMS

Consider the following BVP:

$$x^{\Delta\nabla}(t) = g(t), \ t \in \mathbb{T}_\kappa^\kappa, \tag{3.1}$$

$$x(a) = 0, \tag{3.2}$$

$$x(b) = 0. \tag{3.3}$$

Set

$$h(t) = g(a + b - t), \ t \in [a, \ b].$$

Consider what we shall call the companion BVP,

$$y^{\nabla\Delta}(t) = h(t), \ t \in \mathbb{T}_\kappa^\kappa, \tag{3.4}$$

together with boundary conditions (3.2), (3.3). Let H_1 denote Green's function corresponding to the dynamic boundary value problem (3.1)–(3.3) and H_2 denote Green's function corresponding to the dynamic BVP (3.4), (3.2), (3.3). Atıcı and Guseinov [2] have shown that

$$H_1(t,s) = H_2(t,s) = \begin{cases} \dfrac{(t-a)(s-b)}{b-a}, & t < s, \\ \dfrac{(s-a)(t-b)}{b-a}, & s \le t, \end{cases}$$

where H_i is defined on $\mathbb{T} \times \mathbb{T}_\kappa^\kappa$, $i = 1, 2$. It is proved [2] that x is a solution of (3.1)–(3.3) if and only if

$$x(t) = \int_a^{\rho(b)} H_1(t,s) g(s) \nabla s.$$

Similarly, y is a solution of (3.4), (3.2), (3.3) if and only if

$$y(t) = \int_{\sigma(a)}^{b} H_2(t,s) h(s) \Delta s.$$

Lemma 3.1. *We have the following crossed symmetry properties*

$$\begin{aligned} H_1(t,s) &= H_2(a+b-t, a+b-s), && (t,s) \in \mathbb{T} \times \mathbb{T}_\kappa^\kappa, \\ H_1(a+b-t, s) &= H_2(t, a+b-s), && (t,s) \in \mathbb{T} \times \mathbb{T}_\kappa^\kappa, \\ H_1(t, a+b-s) &= H_2(a+b-t, s), && (t,s) \in \mathbb{T} \times \mathbb{T}_\kappa^\kappa. \end{aligned}$$

Theorem 3.2. *Let x denote the solution of the BVP* (3.1)–(3.3)*, and let y denote the solution of BVP* (3.4)*,* (3.2)*,* (3.3)*. Then*

$$x(a+b-t) = y(t), \quad t \in \mathbb{T}.$$

Proof. According to Lemma 3.1 and Theorem 2.2,

$$\begin{aligned} x(a+b-t) &= \int_a^{\rho(b)} H_1(a+b-t, u) g(u) \nabla u \\ &= \int_a^{\rho(b)} H_2(t, a+b-u) g(u) \nabla u \\ &= \int_{\sigma(a)}^{b} H_2(t,s) g(a+b-s) \Delta s \\ &= \int_{\sigma(a)}^{b} H_2(t,s) h(s) \Delta s = y(t). \end{aligned} \tag{3.5}$$

The proof is complete. □

We say that x and y are cross symmetric if $x(a+b-t) = y(t)$, $t \in \mathbb{T}$.

Corollary 3.3. *Assume g is symmetric; that is, assume $g(t) = h(t)$, $t \in \mathbb{T}$. Assume in addition that $\mathbb{T}$ is the interval of reals, $[a,b]$. Then $x = y$; that is, the solution of* (3.1)–(3.3) *is symmetric.*

Proof. Under the assumptions of Corollary 3.3, the calculations in the proof of Theorem 3.2 reduce to

$$\begin{aligned} x(a+b-t) &= \int_a^b H_1(a+b-t,u)g(u)du \\ &= \int_a^b H_2(t,a+b-u)g(u)du \\ &= \int_a^b H_2(t,s)g(a+b-s)ds \\ &= \int_a^b H_2(t,s)g(s)ds = x(t). \end{aligned}$$

This completes the proof. □

In [4], we show by example that Corollary 3.3 is not valid if $\mathbb{T}$ is a time scale with nonconstant graininess. Suppose one intends to compute a numerical solution for the BVP for the differential equation

$$u''(t) = g(t), \quad t \in [a,b], \tag{3.6}$$

with Dirichlet boundary conditions, (3.2), (3.3) using finite difference methods. Let $\mathbb{T}$ denote a symmetric time scale with nonconstant graininess, and assume $a = \min \mathbb{T}$. Let x_1 denote the solution of (3.1)–(3.3) and let y_1 denote the solution of (3.4), (3.2), (3.3). By Theorem 3.2, x_1 and y_1 are cross symmetric. Consider further the dynamic equations

$$x^{\nabla\Delta}(t) = g(t),\ t \in \mathbb{T}_\kappa^\kappa, \tag{3.7}$$

$$y^{\Delta\nabla}(t) = h(t),\ t \in \mathbb{T}_\kappa^\kappa. \tag{3.8}$$

Let x_2 and y_2 denote the solutions of the respective BVPs, (3.7),(3.2), (3.3) and (3.8), (3.2), (3.3). It is readily shown as in the development of (3.5) that x_2 and y_2 are cross symmetric; that is,

$$x_2(a+b-t) = y_2(t), \quad t \in \mathbb{T}.$$

In particular, BVPs (3.7), (3.2), (3.3) and (3.8), (3.2), (3.3) are companion BVPs. Finally, note that in the case where g is symmetric as well (*i.e.*, $g(t) = h(t)$) then

$$x_1(t) = y_2(t), \quad x_2(t) = y_1(t), \quad t \in \mathbb{T}.$$

Corollary 3.4. *Assume g is symmetric. Let u denote the solution of the BVP* (3.6), (3.2), (3.3). *Let $\mathbb{T}$ denote a symmetric time scale with $a = \min \mathbb{T}$, $b = \max \mathbb{T}$. Further let x_1 denote the solution of the BVP* (3.1), (3.2), (3.3) *and let x_2 denote the solution of the problem* (3.7), (3.2), (3.3). *Assume each of x_1 and x_2 approximate u on $\mathbb{T}$ within a specified tolerance. Then $x = (x_1 + x_2)/2$ approximates u on $\mathbb{T}$ within the specified tolerance and each of u and x are symmetric on their respective domains.*

We now turn our attention to the case where the nonhomogeneous term g is anti-symmetric; that is, we assume that $g(a+b-t) = -g(t)$. Thus, in the above notation, we now assume that $h(t) = -g(t)$. The details in (3.5) readily give the following corollaries.

Corollary 3.5. *Assume g is anti-symmetric; that is, assume $g(t) = -h(t)$, $t \in \mathbb{T}$. Assume in addition that $\mathbb{T}$ is the interval of reals, $[a, b]$. Then $x = -y$; that is, the solution of* (3.6), (3.2), (3.3) *is anti-symmetric.*

Corollary 3.6. *Assume g is anti-symmetric. Let u denote the solution of the BVP* (3.6), (3.2), (3.3). *Let $\mathbb{T}$ denote a symmetric time scale with $a = \min \mathbb{T}$, $b = \max \mathbb{T}$. Let x_1 denote the solution of the BVP* (3.1), (3.2), (3.3) *and let x_2 denote the solution of the BVP* (3.7), (3.2), (3.3). *Assume each of x_1 and x_2 approximate u on $\mathbb{T}$ within a specified tolerance. Then $x = (x_1 + x_2)/2$ approximates u on $\mathbb{T}$ within the specified tolerance and each of u and x are anti-symmetric on their respective domains.*

We briefly turn our attention to arbitrary nonhomogeneous terms, g, and show that such a term can be decomposed into the sum of a symmetric and an anti-symmetric function.

Theorem 3.7. *Let $\mathbb{T}$ be a symmetric time scale. Let g denote a real valued function defined on $\mathbb{T}$. Then there exists a symmetric function, g_1, and an anti-symmetric function, g_2, defined on $\mathbb{T}$ such that $g = g_1 + g_2$.*

Proof. Set $g_1(t) = (g(t) + g(a+b-t))/2$ and $g_2(t) = (g(t) - g(a+b-t))/2$. □

Remark 3.8. Suppose $\mathbb{T}$ is symmetric about $t = 0$. Then the above discussion about symmetry and anti-symmetry becomes a discussion about evenness and oddness.

We close Section 3 with a brief discussion on nonlinear problems and then show how the ideas above apply to a reaction-diffusion partial differential equation. We focus the discussion to the case of symmetry.

Consider the nonlinear dynamic equations

$$x^{\Delta\nabla}(t) = g(t, x(t)), \; t \in \mathbb{T}_\kappa^\kappa, \tag{3.9}$$

$$y^{\nabla\Delta}(t) = h(t, y(t)), \; t \in \mathbb{T}_\kappa^\kappa, \tag{3.10}$$

and assume $h(t, x) = g(a+b-t, x) = g(t, x)$. In particular, assume g satisfies symmetry in t.

Atıcı and Guseinov [2] have shown that x is a solution of the problem (3.9), (3.2), (3.3) if and only if $x \in C(\mathbb{T})$ and

$$x(t) = \int_a^{\rho(b)} H_1(t, s) g(s, x(s)) \nabla s.$$

Similarly, y is a solution of (3.10), (3.2), (3.3) if and only if $y \in C(\mathbb{T})$ and

$$y(t) = \int_{\sigma(a)}^{b} H_2(t,s)h(s,y(s))\Delta s.$$

Employing details completely analogous to the proof of Theorem 3.2, it follows that x is a solution of the BVP (3.9), (3.2), (3.3) if and only if $y(t) = x(a+b-t)$ is a solution of the BVP (3.10), (3.2), (3.3). Hence, precisely the same symmetry and anti-symmetry issues arise for nonlinear dynamic equations.

Let us now consider a heat partial differential equation of the form

$$u_t = \alpha u_{xx} + f(u), \quad x \in [a,b], \quad t > 0,$$

with conditions

$$u(t,a) = 0, \quad u(t,b) = 0, \quad u(0,x) = u_0(x).$$

The above initial-boundary value problems model a number of important natural and physical processes such as quenching, combustion, and pipeline decays in which symmetric solutions are often observed [6]. Apply a finite adaptive algorithm to obtain a partial dynamic equation of the form

$$u_t = \alpha u^{\Delta\nabla} + f(u), \quad x \in \mathbb{T}_\kappa^\kappa, \quad t > 0,$$

with proper Dirichlet boundary conditions. A popular numerical procedure for solving the above problem is to employ a modified Euler's method to obtain an implicit difference scheme of the form

$$(u_{n+1} - u_n)/\Delta t = \alpha u_{n+1}^{\Delta\nabla} + f(u_{n+1}).$$

If we assume that Δt is a constant, then solve algebraically for $u_{n+1}^{\Delta\nabla}$ to obtain the second order dynamic equation

$$u_{n+1}^{\Delta\nabla} = F(u_{n+1}) + g(x, u_n(x)), \tag{3.11}$$

where $g(x, u_n(x)) = u_n(x)/(\alpha\Delta t)$. In particular, we obtain the second order dynamic equation

$$u^{\Delta\nabla} = F(u) + g_n(x).$$

Assume the original shape function, u_0 is symmetric on $[a,b]$ and assume the time scale $\mathbb{T}$ is symmetric. Then the above discussion on symmetry applies. We contend that one should generate symmetric solutions, and so, one should average the cross symmetric solutions of companion boundary value problems.

4. NUMERICAL EXAMPLES

Let

$$\mathbb{T} = \{0, 7/50, 13/50, 9/25, 11/25, 1/2, 14/25, 16/25, 37/50, 43/50, 1\}.$$

It is not difficult to see that the condition (2.9) is satisfied and thus $\mathbb{T}$ is symmetric.

Example 4.1. Consider the following dynamic equation BVP:

$$x^{\Delta\nabla} = g^{(n)}, \qquad x(0) = x(1) = 0, \tag{4.1}$$

$$y^{\nabla\Delta} = h^{(n)}, \qquad y(0) = y(1) = 0, \tag{4.2}$$

where

$$g^{(n)}(t) = t^n,\ h^{(n)}(t) = g^{(n)}(1-t) = (1-t)^n,\ n = 0, 1, 2, \ldots.$$

We observe that as $n = 0$, function $g^{(0)}$ reduces to a constant and therefore the solutions of the pair of companion problems x, y are cross symmetric as we predicted in [4]. The average of solutions $\phi = (x+y)/2$ is symmetric. In a particular situation when all nodes of $\mathbb{T}$ are equally spaced, the solutions x and y become symmetric too. It is also observed in the experiments that for $n > 0$, solutions x and y continue to demonstrate the cross symmetry property, as far as a symmetric time scale $\mathbb{T}$ is used. The function ϕ remains as being symmetric over $\mathbb{T}$.

We demonstrate the above interesting properties from Figure 1 to Figure 4, in which reference values $n = 0, 2, 5, 9$, are used. To show both x and y are in fact good numerical approximations to solutions of the differential equation BVPs

$$u'' = g^{(n)}, \qquad u(0) = u(1) = 0,$$

$$v'' = h^{(n)}, \qquad v(0) = v(1) = 0,$$

we respectively plot both u and v in the same frames of x and y using dotted and dash-dotted curves too.

Example 4.2. Note that

$$g^{(n)} = g_1^{(n)} + g_2^{(n)},\ n = 1, 2, \ldots.$$

where functions

$$g_1^{(n)}(t) = \frac{t^n + (1-t)^n}{2},\ g_2^{(n)}(t) = \frac{t^n - (1-t)^n}{2}$$

are symmetric and anti-symmetric, respectively. We may consider the following pairs of companion dynamic equation BVPs

$$v_1^{\Delta\nabla} = g_1^{(n)}, \qquad v_1(0) = v_1(1) = 0, \tag{4.3}$$

$$v_2^{\nabla\Delta} = g_1^{(n)}, \qquad v_2(0) = v_2(1) = 0, \tag{4.4}$$

and

$$w_1^{\Delta\nabla} = g_2^{(n)}, \qquad w_1(0) = w_1(1) = 0, \tag{4.5}$$

$$w_2^{\nabla\Delta} = g_2^{(n)}, \qquad w_2(0) = w_2(1) = 0. \tag{4.6}$$

These are two pairs of the companion problems as we discussed earlier in Section 3. For $n = 2, 5, 9$, solutions of v_1, v_2 and w_1, w_2 are given in Figures 5, 6, 7, respectively. The solution pairs v_1, v_2 of (4.3), (4.4) demonstrate the fabulous cross symmetry while solutions w_1, w_2 of (4.5), (4.6) show exactly the anti-cross symmetric property studied in the previous section. It is also noticed that the properties remain unchanged as far as $\mathbb{T}$ is symmetric. In a particular case when the nodes of $\mathbb{T}$ are equally spaced, that is, $\mathbb{T}$ possesses a constant graininess, then v_1 is equivalent to v_2 and w_1 is equivalent to w_2. This can be easily verified in computations.

To show that the solutions of the dynamic problems are in fact good approximations to the solutions of the following differential equation BVPs,

$$v'' = g_1^{(n)}, \qquad v(0) = v(1) = 0;$$

$$w'' = g_2^{(n)}, \qquad w(0) = w(1) = 0,$$

we also give solutions v and w in Figures 5–7 in corresponding frames. Needless to say, a refinement of the grids may improve the accuracy of the numerical approximations, but it cannot change the aforementioned cross symmetry or anti-cross symmetry properties of the results.

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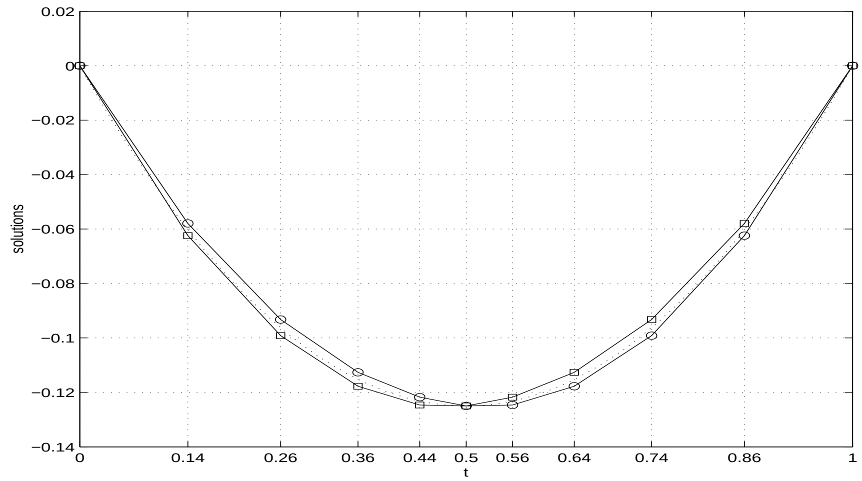

Figure 1. Solutions of the dynamic equation BVPs and the solution of the corresponding differential equation problems. $n = 0$ is used. The circled curve is x, and squared curve is for y. Dotted curve is for u and v.

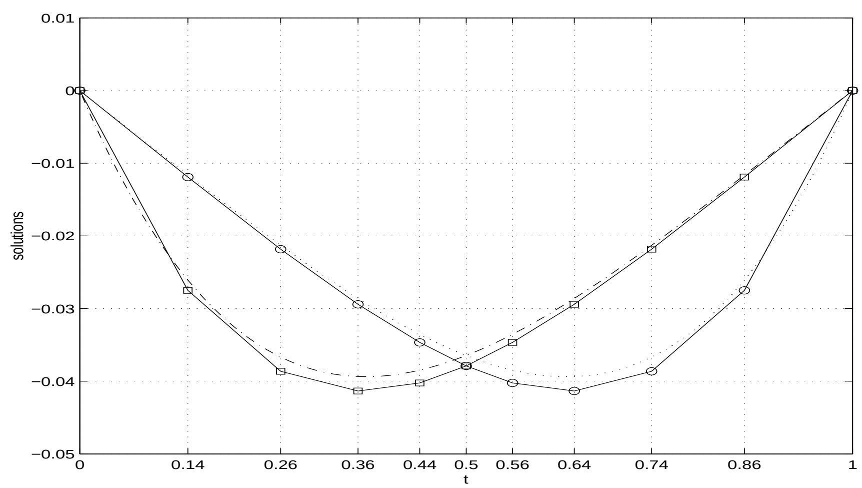

Figure 2. Solutions of the dynamic equation BVPs and the solution of the corresponding differential equation problems. $n = 2$ is used. The circled curve is x, and squared curve is for y. Dotted curve is for u and dash-dotted curve is for v.

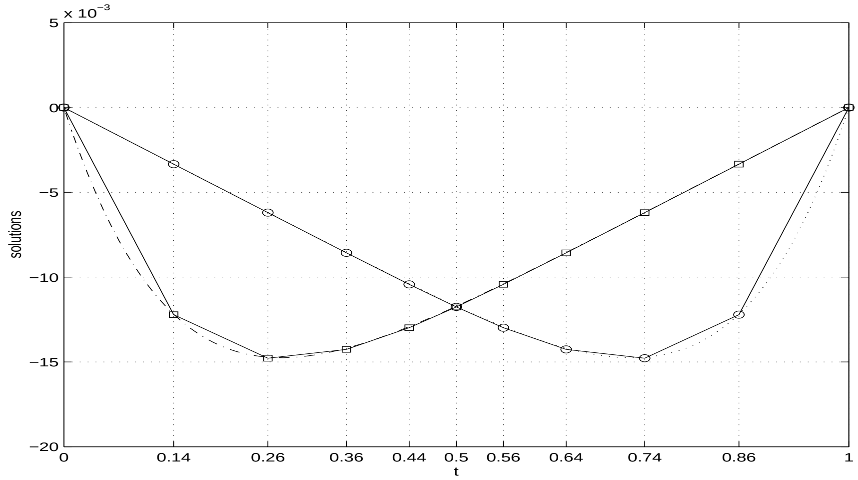

Figure 3. Solutions of the dynamic equation BVPs and the solution of the corresponding differential equation problems. $n = 5$ is used. The circled curve is x, and squared curve is for y. Dotted curve is for u and dash-dotted curve is for v.

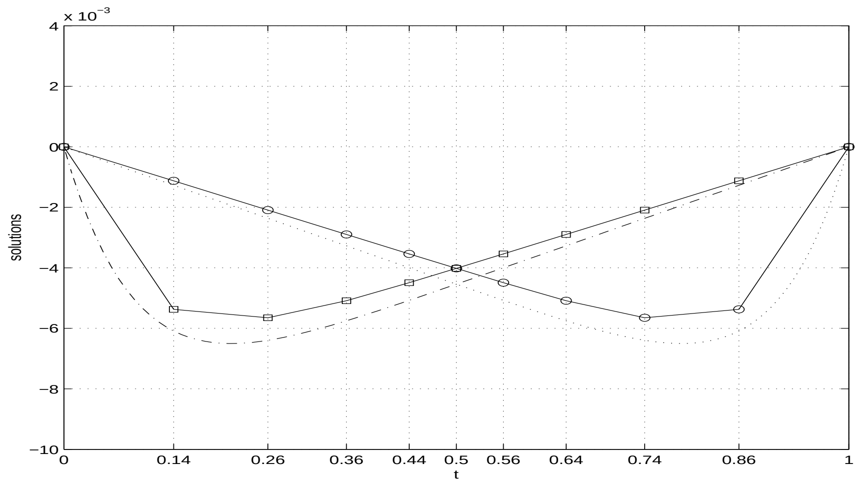

Figure 4. Solutions of the dynamic equation BVPs and the solution of the corresponding differential equation problems. $n = 9$ is used. The circled curve is x, and squared curve is for y. Dotted curve is for u and dash-dotted curve is for v.

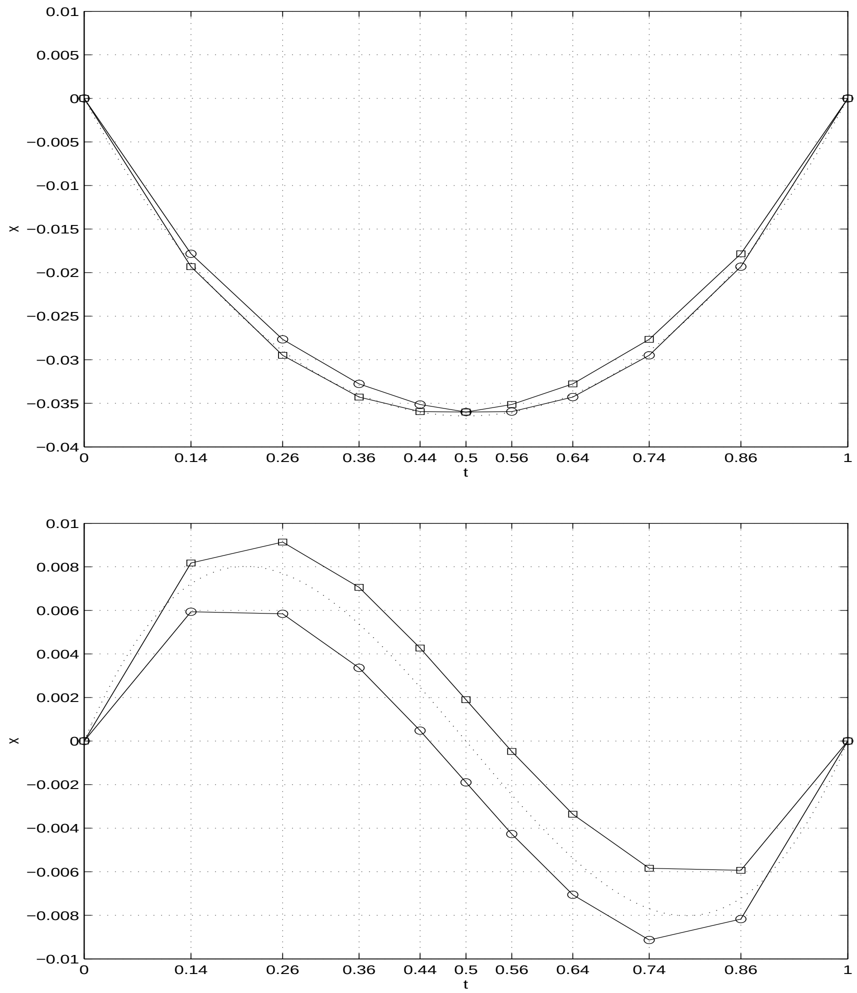

Figure 5. Computed solutions v, v_1 and v_2, (top), and solutions w, w_1 and w_2, (bottom). The circled curves are for v_1 and w_1, while squared curves are for v_2 and w_2. Dotted curves are for v and w. $n = 2$ is considered.

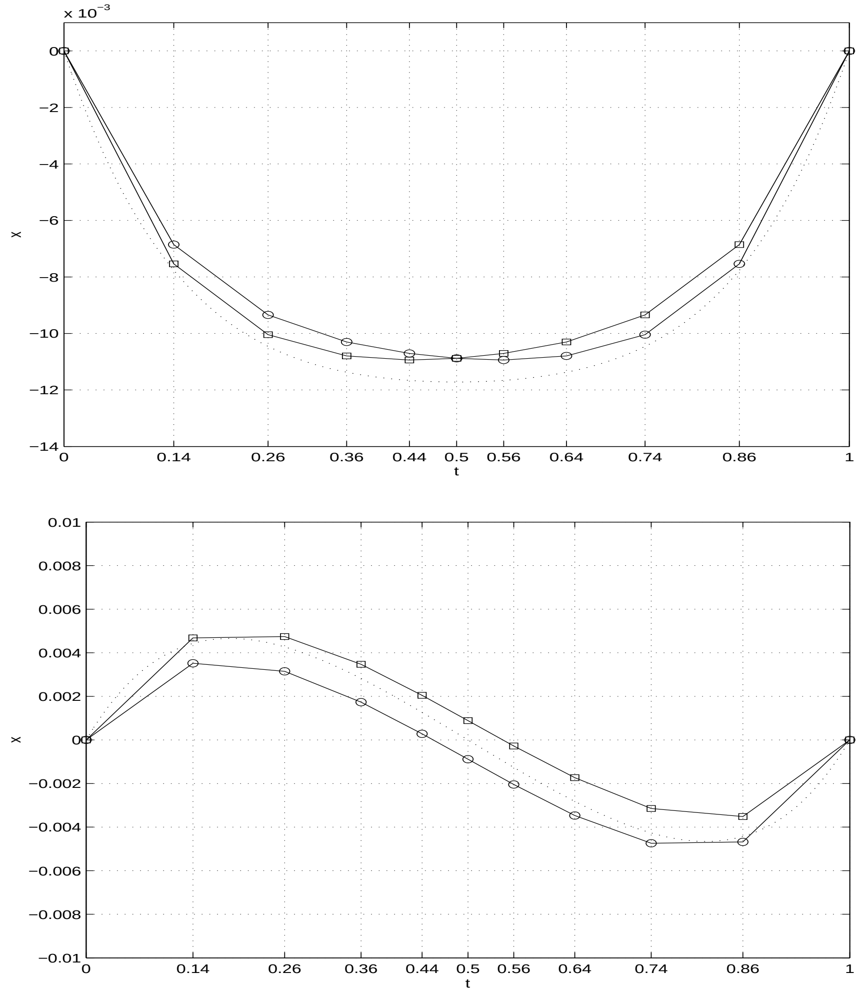

Figure 6. Computed solutions v, v_1 and v_2, (top), and solutions w, w_1 and w_2, (bottom). The circled curves are for v_1 and w_1, while squared curves are for v_2 and w_2. Dotted curves are for v and w. $n = 5$ is considered.

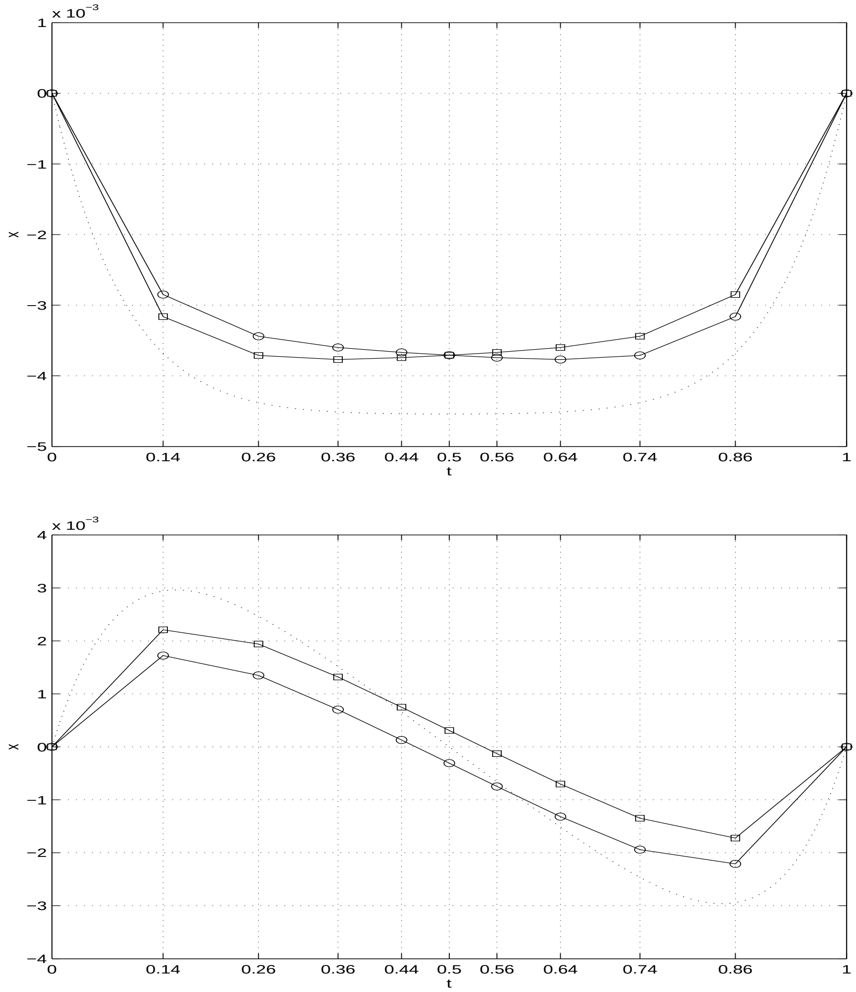

Figure 7. Computed solutions v, v_1 and v_2, (top), and solutions w, w_1 and w_2, (bottom). The circled curves are for v_1 and w_1, while squared curves are for v_2 and w_2. Dotted curves are for v and w. $n = 9$ is used.