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TWO-POINT AND THREE-POINT PROBLEMS FOR FOURTH ORDER DYNAMIC EQUATIONS

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ABSTRACT. Let $\mathbb{T}$ be a closed subset of $\mathbb{R}$ with $\inf \mathbb{T} = -\infty$ and $\sup \mathbb{T} = +\infty$. For the fourth order nonlinear dynamic equation, $u^{\Delta^4} = f(t, u, u^{\Delta}, u^{\Delta^2}, u^{\Delta^3})$, $t \in \mathbb{T}$, where $f : \mathbb{T} \times \mathbb{R}^4 \to \mathbb{R}$ is continuous, we assume solutions of initial value problems are unique and extend to $\mathbb{T}$. We consider questions of the uniqueness of solutions implying the existence of solutions for two-point and for three-point conjugate boundary problems on $\mathbb{T}$.

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1. INTRODUCTION

Let $\mathbb{T}$ be a closed subset of $\mathbb{R}$ with $\inf \mathbb{T} = -\infty$ and $\sup \mathbb{T} = +\infty$. In some of the current literature, $\mathbb{T}$ is called a *time scale* in the context of measure chains; see [1, 2, 3]. For notation, we shall use the convention that for each interval I of $\mathbb{R}$,

$$I_{\mathbb{T}} = I \cap \mathbb{T}.$$

We are concerned with the uniqueness of solutions implying the existence of solutions of certain two-point and three-point boundary value problems for the fourth order dynamic equation on $\mathbb{T}$,

$$y^{\Delta^4} = f(t, y, y^{\Delta}, y^{\Delta^2}, y^{\Delta^3}),\ t \in \mathbb{T}, \tag{1.1}$$

where Δ denotes the delta derivative, and

(A) $f : \mathbb{T} \times \mathbb{R}^4 \to \mathbb{R}$ is continuous.

There is much current activity focused on dynamic equations on time scales, and a good deal of this activity is devoted to boundary value problems. Much of the activity is motivated by Hilger's landmark paper [25] on measure chains in which a unification theory is developed between the continuous calculus and the discrete calculus. Since then, efforts have been made in the context of time scales, in establishing that some results for boundary value problems for ordinary differential equations and their

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discrete analogues are special cases of more general results on measure chains; for a wide variety of problems addressed, see the many references [5–15].

At this point and for the readers' convenience, we state a few definitions which are basic to the calculus on the time scale $\mathbb{T}$ [7, 14, 25]. The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{R}$ is defined by

$$\sigma(t) = \inf\{s > t \mid s \in \mathbb{T}\} \in \mathbb{T}.$$

If $\sigma(t) > t$, t is said to be *right scattered*, whereas, if $\sigma(t) = t$, t is said to be *right dense*. The *backward jump operator* $\rho : \mathbb{T} \to \mathbb{R}$ is defined by

$$\rho(t) = \sup\{s < t \mid s \in \mathbb{T}\} \in \mathbb{T}.$$

If $\rho(t) < t$, t is said to be *left scattered*, and if $\rho(t) = t$, t is said to be *left dense*. If $g : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$, then the *delta derivative of g at t*, $g^{\Delta}(t)$, is defined to be the number (provided it exists), with the property that, given any $\varepsilon > 0$, there is neighborhood U of t, such that

$$|[g(\sigma(t)) - g(s)] - g^{\Delta}(t)[\sigma(t) - s]| \le \varepsilon|\sigma(t) - s|,$$

for all $s \in U$.

In this work, the types of boundary value problems for (1.1), for which we address the question of uniqueness of solutions implying the existence of solutions, would be appropriately called *conjugate* boundary value problems. In particular, given $k \in \{2, 3, 4\}$, $m_1, \dots, m_k \in \mathbb{N}$ such that $\sum_{\ell=1}^{k} m_\ell = 4$, and $t_1 < \cdots < t_k$ in $\mathbb{T}$, we are concerned with solutions of (1.1) satisfying

$$y^{\Delta^j}(t_i) = y_{i,j}, \quad 0 \le j \le m_i - 1, \quad 1 \le i \le k. \tag{1.2}$$

When $k = 4$, the case of uniqueness implies existence for solution of (1.1), (1.2) was studied by Henderson and Yin [24].

For the conjugate boundary value problems for (1.1), our results are motivated by a theorem for conjugate problems for ordinary differential equations proved by Hartman [16] and Klaasen [27], and also by difference equation analogues proved by Henderson [18]. This work does indeed present a unifying result for which the results of [16, 18, 27] are special cases. Other related and notable uniqueness implies existence studies for boundary value problems for both ordinary differential equations and finite difference equations include Davis and Henderson [12] and Henderson [19, 20]. We also need to mention that some questions of uniqueness implies existence for dynamic equations have been addressed by Bohner and Peterson [7], Chyan [8] and Henderson and Yin [23].

Our uniqueness conditions on solutions of boundary value problems for (1.1) will be stated in terms of generalized zeros, whose definition we take from Bohner and Eloe [6].

Definition 1.1. For $y : \mathbb{T} \to \mathbb{R}$, $a \in \mathbb{T}$ is a generalized zero (GZ) of order greater than or equal to k, if either

$$y^{\Delta^j}(a) = 0, \quad j = 0, \dots, k-1,$$

or

(1.3) $$y^{\Delta^j}(a) = 0, \quad j = 0, \dots, k-2, \text{ and } y^{\Delta^{k-1}}(\rho(a))y^{\Delta^{k-1}}(a) < 0.$$

We remark that (1.3) is equivalent to

(1.4) $$y^{\Delta^j}(a) = 0, \quad j = 0, \dots, k-2, \text{ and } (-1)^{k-1}y(\rho(a))y^{\Delta^{k-1}}(a) < 0.$$

In view of this terminology, our uniqueness assumptions on initial value problems and conjugate boundary value problems for (1.1) take the following forms:

(B) Solutions of initial value problems for (1.1) are unique and exist on all of $\mathbb{T}$;

(C) Given $k \in \{2, 3, 4\}$, $m_1, \dots m_k$, such that $\sum_{i=1}^{k} m_i = 4$, and $t_1 < \cdots < t_k$ in $\mathbb{T}$ with $\sigma^{m_\ell - 1}(t_\ell) < t_{\ell+1}$, $1 \le \ell \le k-1$, if $y(t)$ and $z(t)$ are solutions of (1.1) such that $y(t) - z(t)$ has a GZ at t_j of order m_j, $1 \le j \le k$, then $y(t) = z(t)$ on $[t_1, t_k]_{\mathbb{T}}$.

Our final assumption involves a precompactness condition on uniformly bounded sequences of solutions of (1.1).

(D) If $\{y_k(t)\}$ is sequence of solutions of (1.1) for which there exists an interval $[c, d]_{\mathbb{T}}$, with $card[c, d]_{\mathbb{T}} \ge 4$ and there exists an $M > 0$ such that $|y_k(t)| \le M$, for all $t \in [c, d]_{\mathbb{T}}$ and for all $k \in \mathbb{N}$, then there exists a subsequence $\{y_{k_j}(t)\}$ such that $\{y_{k_j}^{\Delta^i}(t)\}$ converges uniformly on $[c, d]_{\mathbb{T}}$, $i = 0, 1, 2, 3$.

This paper can be thought of as a completion of the uniqueness implies existence result for (1.1), (1.2) in the case when $k = 4$ in [24]. In particular, they proved the following

Theorem 1.2. *Assume that with respect to* (1.1), *conditions* (A)–(D) *are satisfied. Then, given any* $t_1 < t_2 < t_3 < t_4$ *in* $\mathbb{T}$ *and any* $y_1, y_2, y_3, y_4 \in \mathbb{R}$, *the boundary value problem* (1.1), *satisfying the boundary conditions*

$$y(t_1) = y_1,\ y(t_2) = y_2,\ y(t_3) = y_3,\ y(t_4) = y_4$$

has a unique solution on $\mathbb{T}$.

In Section 2, we will state for convenience and reference some theorems concerning continuous dependence of solutions of (1.1) on initial conditions and on boundary conditions. Then, in Section 3, we prove that under assumptions (A)–(D), there is a unique solution of the boundary value problem for (1.1) satisfying the two-point and three-point conjugate boundary conditions,

(1.5) $$y^{\Delta^j}(t_i) = y_{i,j}, \quad 0 \le j \le m_i - 1,\ 1 \le i \le k,$$

where $k \in \{2,3\}$, $m_1, \dots, m_k \in \mathbb{N}$ such that $\sum_{\ell=1}^{k} m_\ell = 4$, and $t_1 < \cdots < t_k$ in $\mathbb{T}$. The proof employs shooting methods.

2. CONTINUOUS DEPENDENCE

In this section, we will state theorems concerning the continuous dependence on data of solutions of initial value problems and conjugate boundary value problems for (1.1). We state these results for convenience and reference, and hence will eliminate repeating them later. Our first theorem is from a recent work by Chyan and Yin [11].

Theorem 2.1. *Assume that conditions* (A) *and* (B) *are satisfied. Given a solution* $y(t)$ *of* (1.1) *on* $\mathbb{T}$*, an interval* $[a,b]_{\mathbb{T}}$*, a point* $t_0 \in [a,b]_{\mathbb{T}}$*, and* $\varepsilon > 0$*, there exists a* $\delta(\varepsilon, [a,b]_{\mathbb{T}}) > 0$ *such that, if* $|y^{\Delta^i}(t_0) - z_i| < \delta$*,* $i = 0,1,2,3$*, then there exists a solution* $z(t)$ *of* (1.1) *satisfying* $z^{\Delta^i}(t_0) = z_i$*,* $i = 0,1,2,3$*, and* $|y^{\Delta^i}(t) - z^{\Delta^i}(t)| < \varepsilon$ *on* $[a,b]_{\mathbb{T}}$*,* $i = 0,1,2,3$*.*

In turn, if follows that if (C) is also assumed, then the continuous dependence of solutions on initial conditions, coupled with an application of the Brouwer theorem on invariance of domain, imply that solutions of k-point conjugate problems, $k = 2,3,4$, depend continuously on boundary conditions; for a typical argument, see [26] or [17].

Theorem 2.2. *Assume that with respect to* (1.1)*, conditions* (A)*,* (B) *and* (C) *are satisfied. Given a solution* $y(t)$ *of* (1.1) *on* $\mathbb{T}$*, an interval* $[a,b]_{\mathbb{T}}$*,* $k \in \{2,3,4\}$*,* $m_1, \dots, m_k$*, such that* $\sum_{i=1}^{k} m_i = 4$*, points* $t_1 < \cdots < t_k$ *in* $[a,b]_{\mathbb{T}}$*, with* $\sigma^{m_\ell - 1}(t_\ell) < t_{\ell+1}$*,* $1 \le \ell \le k-1$*, and an* $\varepsilon > 0$*, there exists a* $\delta(\varepsilon, [a,b]_{\mathbb{T}}) > 0$ *such that, if* $|y^{\Delta^j}(t_i) - z_{i,j}| < \delta$*,* $0 \le j \le m_i - 1$*,* $1 \le i \le k$*, then there exists a solution* $z(t)$ *of* (1.1) *satisfying* $z^{\Delta^j}(t_i) = z_{i,j}$*,* $0 \le j \le m_i - 1$*,* $1 \le i \le k$*, and* $|y^{\Delta^i}(t) - z^{\Delta^i}(t)| < \varepsilon$ *on* $[a,b]_{\mathbb{T}}$*,* $i = 0,1,2,3$*.*

3. UNIQUENESS IMPLIES EXISTENCE FOR (1.1), (1.5)

In this section, we prove that hypotheses (A)–(D) imply the existence of solutions of (1.1), (1.5). The method of shooting is employed throughout a numbers of cases.

Theorem 3.1. *Assume that with respect to* (1.1)*, conditions* (A)–(D) *are satisfied. Given* $k \in \{2,3\}$*,* $m_1, \dots, m_k \in \mathbb{N}$ *such that* $\sum_{\ell=1}^{k} m_\ell = 4$*, and* $t_1 < \cdots < t_k$ *in* $\mathbb{T}$*, we are concerned with solutions of* (1.1) *satisfying*

$$y^{\Delta^j}(t_i) = y_{i,j}, \quad 0 \le j \le m_i - 1, \quad 1 \le i \le k.$$

Proof. We will discuss this in several cases.

Case (i): We will discuss three-point boundary conditions.

Subcase (i.1): In this case, we consider the existence of solutions of (1.1) that satisfy the boundary conditions

$$y(t_1) = y_{1,0},\ y^{\Delta}(t_1) = y_{1,1},\ y(t_2) = y_{2,0},\ y(t_3) = y_{3,0} \tag{3.1}$$

or

$$y(t_1) = y_{1,0},\ y(t_2) = y_{2,0},\ y(t_3) = y_{3,0},\ y^{\Delta}(t_3) = y_{3,1}. \tag{3.2}$$

We deal only with the existence of the solution of (1.1) that satisfies the boundary conditions (3.1). Let $z(t)$ be the solution of (1.1) satisfying the initial conditions

$$z(t_1) = y_{1,0},\ z^{\Delta}(t_1) = y_{1,1},\ z^{\Delta^2}(t_1) = 0, z^{\Delta^3}(t_1) = 0.$$

Define

$$S_1 = \{r \in \mathbb{R} \mid \text{ there is a solution } x(t) \text{ of (1.1) with } x(t_1) = z(t_1),$$
$$x^{\Delta}(t_1) = z^{\Delta}(t_1),\ x(t_2) = z(t_2),\ x(t_3) = r\}.$$

Since $z(t_3) \in S_1$, $S_1 \neq \phi$. By similar argument in [24], S_1 is open. We want to show that S_1 is closed.

For contradiction purposes, assume that S_1 is not closed. Then there exists $r_0 \in \overline{S_1} \setminus S_1$ and a strictly monotone sequence $\{r_k\} \subset S_1$ such that $\lim_{k\to\infty} r_k = r_0$. We may assume, without loss of generality, that $r_k \uparrow r_0$. For each $k \in \mathbb{N}$, let $x_k(t)$ be the solution of (1.1) with

$$x_k(t_1) = z(t_1),\ x_k^{\Delta}(t_1) = z^{\Delta}(t_1),\ x_k(t_2) = z(t_2),\ x_k(t_3) = r_k.$$

It follows from (C) that, for each k, $x_k^{\Delta^2}(t_1) \neq x_{k+1}^{\Delta^2}(t_1)$ and $x_k^{\Delta}(t_2) \neq x_{k+1}^{\Delta}(t_2)$. Since $r_{k+1} > r_k$, we have from (C) that for $k \in \mathbb{N}$,

$$x_k(t) < x_{k+1}(t) \text{ on } (t_3, \infty)_{\mathbb{T}}.$$

By the "compactness condition" (D) and the cardinality of $(t_3, \infty)_{\mathbb{T}}$, there exists at least one point $\tau_1 \in (t_3, \infty)_{\mathbb{T}}$ such that

$$x_k(\tau_1) \uparrow +\infty.$$

Now, $\text{Card}(t_1, t_2)_{\mathbb{T}} + \text{Card}(t_2, t_3)_{\mathbb{T}} \geq 1$, otherwise, this is an initial value problem. Again, by the "compactness condition", there exists a point $\tau_2 \in (t_1, t_2)_{\mathbb{T}} \cup (t_2, t_3)_{\mathbb{T}}$ such that either

$$x_k(\tau_2) \downarrow -\infty \text{ if } \tau_2 \in (t_1, t_2)_{\mathbb{T}} \quad \text{or} \quad x_k(\tau_2) \uparrow +\infty \text{ if } \tau_2 \in (t_2, t_3)_{\mathbb{T}}.$$

Now, from the previous results by Henderson and Yin [24] , let $w(t)$ be the solution of (1.1) satisfying the boundary conditions

$$w(t_1) = y_{1,0},\ w(t_2) = y_{2,0},\ w(t_3) = r_0,\ w(\xi) = 0$$

for some $\xi \in (t_3, \infty)_{\mathbb{T}}$ such that $\mathrm{Card}(t_3, \xi)_{\mathbb{T}} \geq 2$. Since $w(t_3) = r_0 > r_k = x_k(t_3)$ for all $k \in \mathbb{N}$, it follows that, for sufficiently large K, $x_K(t) - w(t)$ has GZs at t_1, t_2 and in $(t_3, \tau_1]_{\mathbb{T}}$. Also, $x_K(t) - w(t)$ has a GZ in either $(t_1, t_2]_{\mathbb{T}}$ or $(t_2, t_3)_{\mathbb{T}}$. This is a contradiction to (C). Therefore, S_1 is closed. Consequently, $S_1 = \mathbb{R}$. Choosing $y_{3,0} \in S_1$, we have a solution $\overline{x}(t)$ of (1.1) satisfying

$$\overline{x}(t_1) = z(t_1),\ \overline{x}^{\Delta}(t_1) = z^{\Delta}(t_1),\ \overline{x}(t_2) = z(t_2),\ \overline{x}(t_3) = y_{3,0}.$$

Next define

$$S_2 = \{r \in \mathbb{R} \mid \text{ there is a solution } x(t) \text{ of (1.1) with } x(t_1) = z(t_1), \\ x^{\Delta}(t_1) = z^{\Delta}(t_1),\ x(t_2) = r,\ x(t_3) = y_{3,0}\}.$$

For the solution just produced, $\overline{x}(t_2) \in S_2$ so $S_2 \neq \phi$. Again, by Theorem 2.2, S_2 is an open subset of $\mathbb{R}$. We now claim that S_2 is also closed. Assuming S_2 is not closed, there exists $r_0 \in \overline{S}_2 \setminus S_2$ and a strictly monotone sequence $\{r_k\} \subset S_2$ such that $\lim_{k\to\infty} r_k = r_0$. We may assume again that $r_k \uparrow r_0$, and as before, let $x(t)$ denote the corresponding solution (1.1) with

$$x_k(t_1) = z(t_1),\ x_k^{\Delta}(t_1) = z^{\Delta}(t_1),\ x_k(t_2) = r_k,\ x_k(t_3) = y_{3,0}.$$

Again, by (C), for $k \in \mathbb{N}$, $x_k^{\Delta^2}(t_1) \neq x_{k+1}^{\Delta^2}(t_1)$ and $x_k^{\Delta}(t_3) \neq x_{k+1}^{\Delta}(t_3)$. Using $r_{k+1} > r_k$, (C) and $\mathrm{Card}(t_1, t_2)_{\mathbb{T}} \geq 2$ (or this is an "two-point" problem which we will discuss later), we have that for $k \in \mathbb{N}$,

$$x_k(t) < x_{k+1}(t) \text{ on } (t_1, t_2)_{\mathbb{T}}$$

and

$$x_k(t) > x_{k+1}(t) \text{ on } (t_3, \infty)_{\mathbb{T}}.$$

By (D) and the cardinality of $(t_1, t_2)_{\mathbb{T}}$, there exists $t_1 < \tau_1 < t_2$ and $t_3 < \tau_2$ in $\mathbb{T}$ such that

$$x_k(\tau_1) \uparrow +\infty \text{ and } x_k(\tau_2) \downarrow -\infty.$$

This time, from the result of Henderson and Yin [24], let $w(t)$ be the solution of (1.1) satisfying the boundary conditions,

$$w(t_1) = z(t_1),\ w(t_2) = r_0,\ w(t_3) = z(t_3) = y_{3,0}, w(\xi) = 0$$

for some $\xi \in (t_3, \infty)_{\mathbb{T}}$ such that $\mathrm{Card}(t_3, \xi)_{\mathbb{T}} \geq 2$. Since $w(t_2) = r_0 > r_k = x_k(t_2)$, for all k, it follows that, for sufficiently large K, $x_K(t) - w(t)$ has GZs at t_1 and t_2 and in $(t_3, \tau_2)_{\mathbb{T}}$. Since the $\mathrm{Card}(t_1, t_2)_{\mathbb{T}} \geq 2$, either $x_K(t) - w(t)$ has a GZ in$(t_1, t_2)_{\mathbb{T}}$ or t_1 is a GZ of order 2. Assumption (C) implies that $w(t) = x_K(t)$ on $\mathbb{T}$ which is contradiction. Therefore, S_2 is also closed. If we choose $y_{2,0} \in S_2$, we have a solution $y(t)$ of (1.1) satisfying

$$y(t_1) = y_{1,0},\ y^{\Delta}(t_1) = y_{1,1},\ y(t_2) = y_{2,0},\ y(t_3) = y_{3,0}.$$

This concludes the case (i.1).

Subcase (i.2): In this case, we consider the existence of a solution of (1.1) that satisfies the boundary conditions

$$y(t_1) = y_{1,0},\ y(t_2) = y_{2,0},\ y^{\Delta}(t_2) = y_{2,1},\ y(t_3) = y_{3,0}.$$

We will use the reflexive result from (i.1), that is, let $z(t)$ be a solution of(1.1) satisfying the boundary condition

$$z(\tau) = 0,\ z(t_1) = y_{1,0},\ z(t_2) = y_{2,0},\ z^{\Delta}(t_2) = y_{2,1}$$

where $\tau \in (-\infty, t_1)_{\mathbb{T}}$ and the $\mathrm{Card}(\tau, t_1)_{\mathbb{T}} \geq 2$. Define

$$S = \{r \in \mathbb{R} \mid \text{ there is a solution } x(t) \text{ of } (1.1) \text{ with } x(t_1) = z(t_1) = y_{1,0},$$
$$x(t_2) = z(t_2) = y_{2,0},\ x^{\Delta}(t_2) = y_{2,1},\ x(t_3) = r\}.$$

S is nonempty and open. We want to show that S is closed. Again, we assume the contrary. Then, there exists an $r_0 \in \overline{S} \setminus S$ and a strictly monotone sequence $\{r_k\} \subset S$ such that $\lim_{k\to\infty} r_k = r_0$. We may assume, without loss of generality, that $r_k \uparrow r_0$. For each $k \in \mathbb{N}$, let $x_k(t)$ be the solution of (1.1) with

$$x_k(t_1) = z(t_1),\ x_k(t_2) = z(t_2),\ x^{\Delta}(t_2) = z^{\Delta}(t_2),\ x(t_3) = r_k.$$

It follows from (C), that for each k, $x_k^{\Delta}(t_1) \neq x_{k+1}^{\Delta}(t_1)$ and $x_k^{\Delta^2}(t_2) \neq x_{k+1}^{\Delta^2}(t_2)$. Since $r_{k+1} > r_k$ and $\mathrm{Card}(t_2, t_3)_{\mathbb{T}} \geq 2$ (or this is a two-point BVP which we will discuss later), we have from (D) that there exist $\tau_1, \tau_2 \in \mathbb{T}$ such that $t_2 < \tau_1 < t_3 < \tau_2$ and

$$x_k(\tau_1) \uparrow +\infty \text{ and } x_k(\tau_2) \uparrow +\infty.$$

We know that $\mathrm{Card}(t_1, t_2)_{\mathbb{T}} \geq 1$ (or this is a two-point BVP), and so let $\xi \in \mathbb{T}$ and $t_1 < \xi < t_2$. Again, from the result of Henderson and Yin [24], let $w(t)$ be the solution of (1.1) satisfying the boundary conditions

$$w(t_1) = z(t_1), w(\xi) = z(\xi),\ w(t_2) = z(t_2),\ w(t_3) = r_0.$$

Since $w(t_3) = r_0 > r_k = x(t_3)$ for all $k \in \mathbb{N}$, it follows that for sufficiently large K, $x_K(t) - w(t)$ has GZs at t_1, t_2 and in $(t_3, \tau_2)_{\mathbb{T}}$. Either t_1 is a double GZ or there is a GZ in $(t_1, t_2)_{\mathbb{T}}$. This is a contradiction to (C). Therefore S is closed. Consequently, $S = \mathbb{R}$. Choosing $r = y_{3,0}$, we have a solution $y(t)$ of (1.1) satisfying

$$y(t_1) = z(t_1) = y_{1,0},\ y(t_2) = z(t_2) = y_{2,0},\ y^{\Delta}(t_2) = z^{\Delta}(t_2) = y_{2,1},\ y(t_3) = y_{3,0}.$$

This concludes case (i.2) and case (i).

Case (ii). In this case, we will discuss the two-point problems. Again, there are subcases.

Subcase (ii.1): In this case, we consider the existence of a solution of (1.1) that satisfies the boundary conditions

$$y(t_1) = y_{1,0},\ y^{\Delta}(t_1) = y_{1,1},\ y^{\Delta^2}(t_1) = y_{1,2},\ y(t_2) = y_{2,0}.$$

Let $z(t)$ be the solution of (1.1) satisfying the initial conditions

$$z(t_1) = y_{1,0},\ z^{\Delta}(t_1) = y_{1,1},\ y^{\Delta^2}(t_1) = y_{1,2},\ z^{\Delta^3}(t_1) = 0.$$

Define

$$S = \{r \in \mathbb{R} \mid \text{ there is a solution of (1.1) with } x(t_1) = z(t_1),$$
$$x^{\Delta}(t_1) = z^{\Delta}(t_1),\ x^{\Delta^2}(t_1) = z^{\Delta^2}(t_1),\ x(t_2) = r\}.$$

Since $z(t_2) \in S$, $S \neq \phi$ and S is open. We want to show that S is closed. For contradiction purpose, assume that S is not closed. Then there exists an $r_0 \in \overline{S} \setminus S$ and a strictly monotone sequence $\{r_k\} \subset S$ such that $\lim_{k\to\infty} r_k = r_0$. We may assume, without loss of generality, that $r_k \uparrow r_0$. For each $k \in \mathbb{N}$, let $x_k(t)$ be the solution of (1.1) with

$$x_k(t_1) = z(t_1),\ x_k^{\Delta}(t_1) = z^{\Delta}(t_1),\ x_k^{\Delta^2}(t_1) = x_k^{\Delta^2}(t_1),\ x(t_2) = r_k.$$

It follows from (C) that, for each k, $x_k^{\Delta^3}(t_1) \neq x_{k+1}^{\Delta^3}(t_1)$. Since $r_{k+1} > r_k$, and there are at least three points from $\mathbb{T}$ in $(t_1, t_2)_{\mathbb{T}}$ (otherwise, this is an IVP), we have from (C) that for $k \in \mathbb{N}$,

$$x_k(t) < x_{k+1}(t) \text{ on } (t_1, t_2)_{\mathbb{T}}.$$

By the“compactness condition” (D) and the cardinality of $(t_1, t_2)_{\mathbb{T}}$, there exists a point $\tau_1 \in (t_1, t_2)_{\mathbb{T}}$, $\tau_2 \in (t_2, \infty)_{\mathbb{T}}$ such that

$$x_k(\tau_1) \uparrow +\infty \text{ and } x_k(\tau_2) \uparrow +\infty.$$

Now, from the previous results in case (i), let $w(t)$ be the solution of (1.1) satisfying the boundary conditions

$$w(t_1) = y_{1,0},\ w^{\Delta}(t_1) = y_{1,1},\ w(t_2) = r_0,\ w(\xi) = 0,$$

for some $\xi \in (t_3, \infty)_{\mathbb{T}}$ such that $\text{Card}(t_3, \xi)_{\mathbb{T}} \geq 2$. Since $w(t_2) = r_0 > r_k = x_k(t_2)$ for all $k \in \mathbb{N}$, it follows that, for sufficiently large K, $x_K(t) - w(t)$ has GZs at t_1 with order 2, a GZ in $(t_1, \tau_1]_{\mathbb{T}}$ and a GZ in $(t_2, \tau_2]_{\mathbb{T}}$. This is a contradiction to (C). Therefore, S is closed. Consequently, $S = \mathbb{R}$. Choosing $y_{2,0} \in S$, we have a solution $y(t)$ of (1.1) satisfying

$$y(t_1) = y_{1,0},\ y^{\Delta}(t_1) = y_{1,1},\ y^{\Delta^2}(t_1) = y_{1,2},\ y(t_2) = y_{2,0}.$$

This concludes case (ii.1). The reflexive result of this case, that is, the existence of the solution of (1.1) satisfying the boundary conditions

$$x(t_1) = y_{1,0},\ x(t_2) = y_{2,0},\ x^{\Delta}(t_2) = y_{2,1},\ x^{\Delta^2}(t_2) = y_{2,2}$$

can be obtained similarly.

Subcase (ii.2): In this case, we consider the existence of a solution of (1.1) that satisfies the boundary conditions

$$y(t_1) = y_{1,0},\ y^{\Delta}(t_1) = y_{1,1},\ y(t_2) = y_{2,0},\ y^{\Delta}(t_2) = y_{2,1}.$$

We know that $\mathrm{Card}(t_1, t_2)_{\mathbb{T}} \geq 2$ (otherwise this is an IVP). Using the result from case (i.1), let $z(t)$ be the solution of (1.1) satisfying the boundary conditions

$$z(t_1) = y_{1,0},\ z^{\Delta}(t_1) = y_{1,1},\ z(t_2) = y_{2,0},\ z(\tau) = 0,$$

where $\tau \in (t_2, \infty)_{\mathbb{T}}$ and $\mathrm{Card}(t_2, \tau) \geq 2$. Define

$$S = \{r \in S \mid \text{ there is a solution of (1.1) with } x(t_1) = z(t_1),\\ x^{\Delta}(t_1) = z^{\Delta}(t_1),\ x(t_2) = z(t_2),\ x^{\Delta}(t_2) = r\}.$$

Since $z^{\Delta}(t_2) \in S$, $S \neq \phi$ and S is open. We want to show that S is closed. For contradiction purposes, assume that S is not closed. Then there exist $r_0 \in \overline{S} \setminus S$ and a strictly monotone sequence $\{r_k\} \subset S$ such that $\lim_{k\to\infty} r_k = r_0$. We may assume, without loss of generality, that $r_k \uparrow r_0$. For each $k \in \mathbb{N}$, let $x_k(t)$ be the solution of (1.1) with

$$x_k(t_1) = z(t_1),\ x_k^{\Delta}(t_1) = z^{\Delta}(t_1),\ x(t_2) = z(t_2),\ x_k^{\Delta}(t_2) = r_k.$$

It follows from (C) that, for each k, $x_k^{\Delta^2}(t_1) \neq x_{k+1}^{\Delta^2}(t_1)$. Since $r_{k+1} > r_k$, we have from (C) that for $k \in \mathbb{N}$,

$$x_k(t) > x_{k+1}(t) \text{ on } (t_1, t_2)_{\mathbb{T}} \text{ and } x_k(t) < x_{k+1}(t) \text{ on } (t_2, \infty)_{\mathbb{T}}.$$

By the " compactness condition" (D), there exist $\tau_1 \in (t_1, t_2)_{\mathbb{T}}$ and $\tau_2 \in (t_2, \infty)_{\mathbb{T}}$ such that

$$x_k(\tau_1) \downarrow -\infty \text{ and } x_k(\tau_2) \uparrow +\infty.$$

Now, using the result from case (i), let $w(t)$ be the solution of (1.1) satisfying the boundary conditions

$$w(\xi) = 0,\ w(t_1) = y_{1,0},\ w(t_2) = y_{2,0},\ w^{\Delta}(t_2) = r_0,$$

for some $\xi \in (-\infty, t_1)_{\mathbb{T}}$ such that $\mathrm{Card}(\xi, t_1)_{\mathbb{T}} \geq 2$. Since $w^{\Delta}(t_2) = r_0 > r_k = x_k^{\Delta}(t_2)$ for all $k \in \mathbb{N}$, it follows that, for sufficiently large K, $x_K(t) - w(t)$ has GZs at t_1 and t_2, and GZs in $(t_1, \tau_1]_{\mathbb{T}}$ and $(t_2, \tau_2]_{\mathbb{T}}$. This is a contradiction to (C). Therefore, S is closed. Consequently, $S = \mathbb{R}$. Choosing $y_{2,1} \in S$, we have a solution $y(t)$ of (1.1) satisfying

$$y(t_1) = y_{1,0},\ y^{\Delta}(t_1) = y_{1,1},\ y(t_2) = y_{2,0},\ y^{\Delta}(t_2) = y_{2,1}.$$

This concludes case (ii.2). The proof is complete. □

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