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FIXING THE LADDER OPERATORS ON TWO TYPES OF HERMITE FUNCTIONS

KRISTINE EY AND ANDREAS RUFFING

Munich University of Technology, Dept. of Mathematics, Boltzmannstraße 3,
D–85747 Garching, Germany, EU. *E-mail:* ey@appl-math.tu-muenchen.de

Munich University of Technology, Dept. of Mathematics, Boltzmannstraße 3,
D–85747 Garching, Germany, EU. *E-mail:* ruffing@appl-math.tu-muenchen.de

ABSTRACT. Hermite Functions and their generalizations are addressed within the framework of time scale calculus on two different time scale structures. A ladder operator formalism is presented in both cases. The main focus of the article is elucidated: The ladder operator formalism is exhibited as a tool for determining orthogonality measures to recursive systems of polynomials. This procedure is performed in the case of $\mathbb{T} = \mathbb{R}$ and in the case of $\mathbb{T}$ being the closure of a q-lattice with $\mathbb{Z}$-grading. The main differences between the continuous and the discrete scenario are worked out. From the spectral theoretical viewpoint, the point spectra of those symmetric operators are determined to which the Hermite functions resp. their generalizations are eigenfunctions.

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1. A GENERAL INTRODUCTION

Orthogonal Polynomials of a discrete variable have been subject to a lot of investigations throughout the last decades, see [13]. Several of the classified orthogonal polynomials in context of the Askey–Wilson scheme fit into the context of basic hypergeometric functions which can be considered as q-generalizations of the hypergeometric functions, see for instance [7, 11]. Let us first give a short overview about the origins of q-special functions and let us reflect some of the related historical developments in analysis.

Leonhard Euler gave in 1748 a celebrated formula for the number $p(n)$ of partitions of a positive integer n into positive integers, where the decomposition consists of decreasing numbers in the sum without any repetitions:

$$\prod_{k=0}^{\infty} (1 - q^{k+1})^{-1} = \sum_{n=0}^{\infty} p(n)\, q^n \tag{1.1}$$

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while the parameter q is ranging in the open interval $(0, 1)$. Here occur the later on called "quanta" q^n. Deriving all information on a demanded function – in this example it is $p(n)$ – from a given generating function reflects a general concept within the analysis of special functions. In this particular example, the role of the generating function is played by the expression $\prod_{k=0}^{\infty}(1-q^{k+1})^{-1}$ on the left hand side of (1.1).

In his "Disquisitiones Generales Circa Seriem Infinitam" from 1813, Carl-Friedrich Gauss gives a fundamental concept for analyzing special functions, namely the structure of hypergeometric functions. Later on, the concept was generalized by Eduard Heine in the context of basic hypergeometric functions. These developments of systematic approaches to the understanding of special functions had a deep impact on mathematical analysis in general. Several types of special functions could be analyzed in a new way when using the Gauss–Heine approach. Let us for instance mention three classical examples of orthogonal polynomials, namely Hermite polynomials, Laguerre polynomials and Legendre polynomials which can be addressed in the light of the hypergeometric Gauss approach.

An alternative to understanding many orthogonal polynomials lies again in the use of suitable generating functions. For example, in the case of the classical continuous Hermite polynomials $H_n(x)$, where $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$, the generating function defines the polynomials $H_n(x)$ by the equation

$$e^{2xy-y^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!}\, y^n, \qquad (x, y) \in \mathbb{R}^2. \tag{1.2}$$

The orthogonality property of a suitable given sequence of polynomials is closely confined to the orthogonality support and its structure. Concerning the support, not only simple structures shall be admitted but one is also interested in general results on more complicated subsets of the real axis or of any real linear space. The need for including more difficult spatial patterns stands in a long, general tradition:

Bernhard Riemann pointed out in the last part of his famous habilitation speech from 1854, "Über die Hypothesen, welche der Geometrie zugrunde liegen":

Die Frage über die Gültigkeit der Voraussetzungen der Geometrie im Unendlichkleinen hängt zusammen mit der Frage nach dem innern Grund der Maßverhältnisse des Raumes. Bei dieser Frage, welche wohl noch zur Lehre vom Raume gerechnet werden darf, kommt die obige Bemerkung zur Anwendung, daß bei einer diskreten Mannigfaltigkeit das Prinzip der Maßverhältnisse schon in dem Begriffe dieser Mannigfaltigkeit enthalten ist, bei einer stetigen aber anders woher hinzukommen muß. Es muß also entweder das dem Raume zugrunde liegende Wirkliche eine diskrete Mannigfaltigkeit bilden, oder der Grund der Maßverhältnisse außerhalb, in darauf wirkenden bindenden Kräften gesucht werden."

In particular, these words reveal the need of a **unified and extended approach** between discrete and continuous analytic structures. Until now, this has a great meaning to applications not only in mathematics but also within fundamental theories of physics, like general relativity or quantum theory.

So far the historical embedding of approaches linking structures from continuous and discrete analysis. In recent years, a fruitful and promising approach towards unifications and extensions within the theory of differential resp. difference equations has been given by the so-called **Time Scale Approach** which was originally created by Stefan Hilger [8, 3]. This approach will provide the basic mathematical framework for our investigations throughout this article.

2. BASIC DEFINITIONS AND SCOPE OF THE ARTICLE

In this section, we review some of the basic definitions in time scale analysis where we refer to [6]. Moreover, we introduce the ladder operator concept for an orthogonal polynomial system on a given time scale $\mathbb{T}$.

Definition 2.1. Let $\mathbb{T} \subseteq \mathbb{R}$ be a time scale, i.e., an arbitrary nonempty closed subset of the real numbers. For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is given by

$$\sigma(t) := \inf \{ s \in \mathbb{T} \mid s > t \}$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup \{ s \in \mathbb{T} \mid s < t \}.$$

The graininess function of $\mathbb{T}$ is defined by

$$\mu : \mathbb{T} \to [0, \infty], \ t \mapsto \mu(t) := \sigma(t) - t$$

and for a given function $f : \mathbb{T} \to \mathbb{R}$, we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ and respectively the function $f^\rho : \mathbb{T} \to \mathbb{R}$ by the conventions

$$\forall t \in \mathbb{T} : \qquad f^\sigma(t) = f(\sigma(t)), \qquad f^\rho(t) = f(\rho(t)).$$

Finally, we denote

$$L^2(\mathbb{T}) := \left\{ f : \mathbb{T} \to \mathbb{C} \ \Big| \ \int_{\mathbb{T}} f(t)\, \overline{f(t)}\, d\mu < \infty \right\} \tag{2.1}$$

consisting of all square integrable functions on the time scale $\mathbb{T}$ where $d\mu$ shall denote a suitable weighted Lebesgue integration measure.

Let us look for instance at an example for the space $L^2(\mathbb{T})$: Let $\mathbb{T} = \mathbb{R}$. Then

$$L^2(\mathbb{T}) = L^2(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} \ \Big| \ \int_{\mathbb{R}} f(t)\, \overline{f(t)}\, dt < \infty \right\}. \tag{2.2}$$

In this article, we will basically be concerned with a so-called ladder operator concept for the continuous resp. a particular discrete version of the Hermite polynomials. Let us first give the following general definition of an **Orthogonal Polynomial System** on $\mathbb{T}$.

Definition 2.2. Let $\mathbb{T} \subseteq \mathbb{R}$ be a time scale and let $(P_n)_{n\in\mathbb{N}_0}$ be a sequence of polynomials where $P_n : \mathbb{T} \to \mathbb{R}$ for any $n \in \mathbb{N}_0$. Let $(a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0}, (c_n)_{n\in\mathbb{N}_0}, (d_n)_{n\in\mathbb{N}_0}$ be sequences of real numbers, such that

$$a_n\, P_{n+1}(x) + b_n\, xP_n(x) + c_n\, P_{n-1}(x) = d_n\, P_n(x) \tag{2.3}$$

for every $x \in \mathbb{T}$ where $n \in \mathbb{N}_0$ and $c_0 = 0$, moreover let

$$\forall x \in \mathbb{T}: \quad P_0(x) = 1, \;\; a_0P_1(x) + b_0x = d_0 \tag{2.4}$$

$$\forall n \in \mathbb{N}: \quad a_nc_{n+1} > 0, \;\; b_n \neq 0, \qquad a_0 > 0, \qquad c_0 = 0. \tag{2.5}$$

We refer to the sequence $(P_n)_{n\in\mathbb{N}_0}$ by the name **Orthogonal Polynomial System** or simply **OPS** on the time scale $\mathbb{T}$.

The expression "Orthogonal Polynomial System" is motivated according to Favard's theorem, see [7], which establishes a close connection between three-term reccurence relations of suitable polynomials and corresponding orthogonality measures.

We now define the ladder operators for an orthogonal polynomial system on $\mathbb{T}$.

Definition 2.3. Let $(P_n)_{n\in\mathbb{N}_0}$ be an orthogonal polynomial system on a given time scale $\mathbb{T}$ and $U \subseteq L^2(\mathbb{T})$ a dense domain. Let moreover the linear operators $a, a^+ : U \to L^2(\mathbb{T})$ be fixed by the equation

$$(af, g) = (f, a^+g) \tag{2.6}$$

for all $f, g \in U$, $(\circ, \circ)$ denoting the canonical standard scalar product in $L^2(\mathbb{T})$ which is induced by (2.1). We refer to the operators a, a^+ by the name **Ladder Operators** acting on the OPS $(P_n)_{n\in\mathbb{N}_0}$ if there exist sequences of real numbers $(\alpha_n)_{n\in\mathbb{N}_0}, (\beta_n)_{n\in\mathbb{N}_0}$ and a function $\psi \in U$ such that $a\psi = 0$, $\beta_0 = 0$ as well as

$$\forall n \in \mathbb{N}_0: \quad a^+(P_n\psi) = \alpha_n\, P_{n+1}\psi \in L^2(\mathbb{T}), \quad a(P_n\psi) = \beta_n\, P_{n-1}\psi \in L^2(\mathbb{T}) \tag{2.7}$$

in combination with

$$\forall m, n \in \mathbb{N}_0: \quad \int_{\mathbb{T}} P_m(x)\, P_n(x)\, \psi(x)\, \overline{\psi(x)}\, d\mu = \nu_n^2\, \delta_{mn}, \tag{2.8}$$

$(\nu_n^2)_{n\in\mathbb{N}_0}$ being a sequence of positive numbers, relation (2.8) hence yielding an orthogonality condition on the polynomials under consideration.

In this article we address the examples of ladder operators on the following two OPS:

1. The continuous classical Hermite polynomials, being introduced by

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0, \qquad n \in \mathbb{N}_0, \qquad x \in \mathbb{T} = \mathbb{R} \tag{2.9}$$

with initial conditions $H_0(x) = 1$ and $H_1(x) = 2x$. What are their ladder operators?

2. Generalizations of the continuum Hermite polynomials, given for $n \in \mathbb{N}_0$ as follows:

$$H^q_{n+1}(x) - \alpha x q^n H^q_n(x) + \alpha\, \frac{1-q^n}{1-q}\, H^q_{n-1}(x) = 0, \qquad x \in \mathbb{T} = \overline{\{\beta q^{\mathbb{Z}}, -\beta q^{\mathbb{Z}}\}} \tag{2.10}$$

where $\alpha, \beta > 0$, $q > 1$, the initial conditions being provided by $H^q_0(x) = 1$ and $H^q_1(x) = \alpha x$. Can one construct suitable ladder operators?

In Section 3, the ladder operator problem for the continuous Hermite polynomials (situation 1) is exhibited in the case $\mathbb{T} = \mathbb{R}$. In Section 4, the ladder operator problem is addressed for $q > 1$ (situation 2). In Section 5, we give a summary from the viewpoint of spectral theory for linear operators.

3. CONTINUOUS CLASSICAL HERMITE POLYNOMIALS

Let us first state the objects of Definition 2.3 for the particular case $\mathbb{T} = \mathbb{R}$.

Let the OPS $(P_n)_{n\in\mathbb{N}_0}$ be given by the $(H_n)_{n\in\mathbb{N}_0}$ from (2.9). Next we specify the set $U \subseteq L^2(\mathbb{R})$. Let $n \in \mathbb{N}$ and $\lambda > 0$. The set $G(\lambda, n)$ shall then consist of the functions

$$\psi : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \psi(x) := \sum_{j=0}^{k} c_j\, x^j\, e^{-\lambda x^{2n}} \tag{3.1}$$

where $k \in \mathbb{N}_0$ and $c_j \in \mathbb{C}$. Let now $U \subseteq L^2(\mathbb{R})$ be the $\mathbb{C}$-algebra $\mathbb{G}$ which is generated by all possible sets $G(\lambda, n)$ (the symbol $\mathbb{G}$ shall stand for "Gaussian bell-shaped functions"):

$$U := \mathbb{G} = \{fg, f+g \mid \exists\, \mu, \nu > 0,\ m, n \in \mathbb{N} :\ \ f \in G(\mu, m),\ g \in G(\nu, n)\}. \tag{3.2}$$

By construction, the set U is dense in $L^2(\mathbb{R})$ as it contains the finite complex span of all functions $v_n : \mathbb{R} \to \mathbb{R}$, $x \mapsto v_n(x) := x^n e^{-\frac{1}{2}x^2}$ where $n \in \mathbb{N}_0$. To specify a^+ and a, we first introduce the operators $D : U \to U$ and $X : U \to U$ by

$$(Df)(x) := f'(x), \qquad (Xf)(x) := xf(x), \qquad x \in \mathbb{R}, \qquad f \in U. \tag{3.3}$$

We now define the operators $a : U \to U$ and $a^+ : U \to U$ resp. the rescaled versions $A, A^+ : U \to U$ by

$$a := D + X, \quad a^+ := -D + X, \quad A := \frac{1}{\sqrt{2}}\,(D+X), \quad A^+ := \frac{1}{\sqrt{2}}\,(-D+X). \tag{3.4}$$

Integration by parts shows that the operators a and a^+ resp. A and A^+ indeed obey a relation of type (2.6) for any $f, g \in U$, $(\circ, \circ)$ denoting the standard scalar product in $L^2(\mathbb{R})$.

Theorem 3.1. *Let* $\psi : \mathbb{R} \to \mathbb{R}$, $x \mapsto \psi(x) := e^{-\frac{1}{2}x^2}$ *and* a, a^+ *be given by* (3.4). *The operators* a, a^+ *are ladder operators on the OPS* $(H_n)_{n\in\mathbb{N}_0}$*, i.e., in particular*

$$a\psi = 0, \tag{3.5}$$

$$\forall m,n \in \mathbb{N}_0 : \quad \int_{\mathbb{R}} H_m(x)\, H_n(x)\, \psi(x)\, \overline{\psi(x)}\, dx = \nu_m^2\, \delta_{mn}, \tag{3.6}$$

$$\forall n \in \mathbb{N}_0 : \quad a^+(H_n\psi) = H_{n+1}\psi, \tag{3.7}$$

$$\forall n \in \mathbb{N}_0 : \quad a(H_n\psi) = 2n\, H_{n-1}\psi. \tag{3.8}$$

Proof. The first equation, (3.5), is evident.

Using the operators A, A^+ we are next going to verify the orthogonality relation (3.6). To do so, one easily derives the commutation relation

$$\forall f \in U : \qquad AA^+f - A^+Af = f \tag{3.9}$$

from the commutation relation $\forall f \in U :\ DXf - XDf = f$. We claim the following fact which reveals an eigenvalue situation of the operator $A^+A : U \to U$:

$$A^+A\,(A^+)^n\psi = n\,(A^+)^n\psi, \qquad n \in \mathbb{N}_0 \tag{3.10}$$

where $\psi : \mathbb{R} \to \mathbb{R},\ x \mapsto \psi(x) := e^{-\frac{1}{2}x^2}$. (3.10) is obviously true for $n = 0$. By induction, using (3.9), statement (3.10) can be proven. For $n \in \mathbb{N}_0$ we abbreviate $\varphi_n := (a^+)^n\psi$. Hence it follows for $m, n \in \mathbb{N}_0$ with $m \neq n$:

$$(A^+A\varphi_m, \varphi_n) = (\varphi_m, A^+A\varphi_n) \quad \Leftrightarrow \quad m\,(\varphi_m, \varphi_n) = n\,(\varphi_m, \varphi_n) \tag{3.11}$$

where we have twice made use of the property $\forall f, g \in U : (af, g) = (f, a^+g)$ resp. $\forall f, g \in U : (Af, g) = (f, A^+g)$. As $m \neq n$, (3.11) directly implies

$$\int_{\mathbb{R}} \varphi_m(x)\, \varphi_n(x)\, dx = \int_{\mathbb{R}} H_m(x)\, H_n(x)\, (\psi(x))^2\, dx = 0 \tag{3.12}$$

which yields the orthogonality statement (3.6) (note that ψ is real-valued).

We address now (3.7) as follows: First, one can check by direct calculation that $a^+Xf - Xa^+f = -f$ for all $f \in U$. Moreover, the function ψ fulfills the equation

$$a^+\psi - 2X\psi = 0. \tag{3.13}$$

Applying successively the operator a^+ to equation (3.13), one arrives by induction at:

$$(a^+)^{n+1}\psi - 2X\,(a^+)^n\psi + 2n\,(a^+)^{n-1}\psi = 0, \qquad n \in \mathbb{N}_0. \tag{3.14}$$

Note that the actions of a^+ on the considered objects are well defined as they all live in U. Comparing (3.14) with the definition of the OPS $(H_n)_{n\in\mathbb{N}_0}$ from (2.9), we arrive at claim (3.7).

Let us finally verify statement (3.8). We start from recurrence relation (3.14) which is now rewritten as

$$\varphi_{n+1}(x) - 2x\varphi_n(x) + 2n\varphi_{n-1}(x) = 0, \qquad n \in \mathbb{N}_0, \qquad x \in \mathbb{R} \tag{3.15}$$

where we impose for $x \in \mathbb{R}$ the initial conditions $\varphi_0(x) = e^{-\frac{1}{2}x^2}$ and $\varphi_1(x) = 2x\varphi_0(x)$. As all the functions φ_n are in $L^2(\mathbb{R})$, we can ask for their normalized versions $(\psi_n)_{n\in\mathbb{N}_0}$ such that $(\psi_i, \psi_j) = \delta_{ij}$ for all $i, j \in \mathbb{N}_0$. For $n \in \mathbb{N}_0$ we denote $\nu_n := \sqrt{(\varphi_n, \varphi_n)}$. Introducing the abbreviations $a_n := \frac{\nu_{n+1}}{2\nu_n}$ and $b_n := \frac{n\nu_{n-1}}{\nu_n}$, we derive from (3.15):

$$X\psi_n = a_n\, \psi_{n+1} + b_n\, \psi_{n-1}, \qquad n \in \mathbb{N}_0 \tag{3.16}$$

where we have chosen the operator notation. Note that X is a symmetric operator on U, i.e., $\forall f, g \in U : (Xf, g) = (f, Xg)$ implying $\forall m, n \in \mathbb{N}_0 : (X\psi_m, \psi_n) = (\psi_m, X\psi_n)$ which requires $\forall n \in \mathbb{N}_0 :\ a_n = b_{n+1}$. This leads to the recurrence relation

$$\forall n \in \mathbb{N}_0 :\ \nu_{n+1} = \sqrt{2n+2}\, \nu_n, \qquad \nu_0 = \sqrt{\int_{\mathbb{R}} e^{-x^2}\, dx} = \pi^{\frac{1}{4}}. \tag{3.17}$$

Due to the definition of the functions $\psi_n\ (n \in \mathbb{N}_0)$, it follows from (3.7) that

$$\forall n \in \mathbb{N}_0 :\ A^+\psi_n = \sqrt{n+1}\, \psi_{n+1}. \tag{3.18}$$

As $a = -a^+ + 2X$ and due to (3.16) we conclude that both, a and a^+ map the finite span $\{\ \psi = \sum_{k=0}^{n} c_k\ \psi_k \mid c_k \in \mathbb{C},\ n \in \mathbb{N}_0\ \} \subseteq U$ into itself. As a consequence, we obtain that the equality

$$\forall m, n \in \mathbb{N}_0 :\quad (A^+\psi_m, \psi_n) = (\psi_m, A\psi_n) \tag{3.19}$$

uniquely fixes the action of A on the functions ψ_n by

$$\forall n \in \mathbb{N}_0 :\ A\psi_n = \sqrt{n}\, \psi_{n-1} \ \Leftrightarrow\ A\, \frac{\varphi_n}{\nu_n} = \sqrt{n}\, \frac{\varphi_{n-1}}{\nu_{n-1}} \ \Leftrightarrow \tag{3.20}$$

$$\forall n \in \mathbb{N}_0 :\ a\varphi_n = 2n\, \varphi_{n-1} \ \Leftrightarrow\ a(H_n\psi) = 2n\, H_{n-1}\psi. \tag{3.21}$$

This proves (3.8). Hence Theorem 3.1 is verified in total. □

As for the concept of ladder operators, the reader is also invited to consider for instance the articles [2, 5, 12, 14]. In the sequel, we will investigate the concept of ladder operators in context of discrete modifications of the Hermite polynomials which are closely related to the discrete q-Hermite polynomials of type I, see [4, 9, 11].

4. MODIFIED DISCRETE HERMITE POLYNOMIALS

We now address the polynomials

$$H^q_{n+1}(x) - \alpha x q^n H^q_n(x) + \alpha\, \frac{1-q^n}{1-q}\, H^q_{n-1}(x) = 0, \qquad x \in \mathbb{T} := \overline{\{\beta q^{\mathbb{Z}}, -\beta q^{\mathbb{Z}}\}} \tag{4.1}$$

where $n \in \mathbb{N}_0, \alpha, \beta > 0, q > 1$, the initial conditions being provided by $H^q_0(x) = 1$ and $H^q_1(x) = \alpha x$. We want to elucidate how the ladder operator formalism helps us to find an orthogonality measure to these polynomials being concentrated on a suitable support $\Omega \subseteq \mathbb{T}$. To do so, we are going to prepare the tools by

Definition 4.1. Let $\mathbb{T} = \overline{\{\beta q^{\mathbb{Z}}, -\beta q^{\mathbb{Z}}\}}$ with a fixed number $\beta > 0$. We refer to the Hilbert space $\mathbb{J}$ of all Jackson square integrable functions by the set of all functions $f : \mathbb{T} \to \mathbb{C}$ fulfilling the property

$$(f, f) := (q-1) \sum_{j=-\infty}^{\infty} \beta q^j \left(f(\beta q^j)\overline{f(\beta q^j)} + f(-\beta q^j)\overline{f(-\beta q^j)} \right) \tag{4.2}$$

hence corresponding to the set $L^2(\mathbb{T})$. A standard scalar product in $\mathbb{J}$ is given by

$$(f, g) := (q-1) \sum_{j=-\infty}^{\infty} \beta q^j \left(f(\beta q^j)\overline{g(\beta q^j)} + f(-\beta q^j)\overline{g(-\beta q^j)} \right) \tag{4.3}$$

for all complex-valued $\mathbb{J}$-functions over $\mathbb{T}$. $B := \{e_m^\mu \mid m \in \mathbb{Z},\ \mu \in \{+1, -1\}\} \subseteq \mathbb{J}$ shall be the set of all functions with $(e_m^\mu, e_n^\nu) = \delta_{mn}\delta_{\mu\nu}$ and $e_m^\mu(\nu\beta q^n) = \frac{1}{\sqrt{\beta}\sqrt{q-1}}\,\delta_{mn}\delta_{\mu\nu}$ where $m, n \in \mathbb{Z}$ and $\mu, \nu \in \{+1, -1\}$. Moreover, we introduce for suitable $f : \mathbb{T} \to \mathbb{C}$ and for any $x \in \mathbb{T} \setminus \{0\}$:

$$(D_q f)(x) := \frac{f(qx) - f(x)}{qx - x}, \qquad (R_q f)(x) := f(qx), \qquad (L_q f)(x) := f(q^{-1}x), \tag{4.4}$$

additionally $(D_q f)(0) := \lim_{x\to 0}(D_q f)(x)$, $(R_q f)(0) := f(0)$, $(L_q f)(0) := f(0)$. Following a notation in the general literature on q-special functions, we will refer to the scalar product (f, g) of two $\mathbb{J}$-functions from (4.3) also by the symbol

$$\int_{\mathbb{T}} f(x)\,\overline{g(x)}\,d_q x \ := \ (q-1) \sum_{j=-\infty}^{\infty} \beta q^j \left(f(\beta q^j)\overline{g(\beta q^j)} + f(-\beta q^j)\overline{g(-\beta q^j)} \right). \tag{4.5}$$

Wanted are thus the following objects with suitable sequences $(\alpha_n)_{n\in\mathbb{N}_0}, (\beta_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{R}$ and $(\nu_m^2)_{m\in\mathbb{N}_0} \subseteq \mathbb{R}^+$:

$$\varphi : \mathbb{T} \to \mathbb{R} : \int_{\mathbb{T}} H_m^q(x) H_n^q(x)\,(\varphi(x))^2\,d_q x = \nu_m^2\,\delta_{mn}, \qquad m, n \in \mathbb{N}_0, \tag{4.6}$$

$$a_q, a_q^+ : U \to U \quad : \quad a_q^+(H_n^q\varphi) = \alpha_n\,H_{n+1}^q\varphi, \quad a_q(H_n^q\varphi) = \beta_n\,H_{n-1}^q\varphi \tag{4.7}$$

satisfying a formal adjointness relation on a maximal common dense domain $U \subseteq \mathbb{J}$,

$$\forall f, g \in U : (a_q f, g) = (f, a_q^+ g) \tag{4.8}$$

in the sense of the general Definition 2.3.

Theorem 4.2. *Let $a_q, a_q^+ : U \to U$ on a maximal common dense domain $U \subseteq \mathbb{J}$ be given as follows:*

$$a_q^+ := -D_q + \xi\,R_q, \qquad a_q := q^{-1}\,(L_q D_q + L_q\,\xi) \tag{4.9}$$

where ξ is a fixed function $\xi : \mathbb{T} \to \mathbb{R}$. A sufficient criterion on a_q, a_q^+ for being ladder operators to the polynomials (4.1) *is the following particular structure of $\varphi = \psi$ from Definition 2.3: Let $\varphi \neq 0$ be a non-negative $\mathbb{J}$-function, being recursively given by*

$$\varphi^2(qx) = (1 + \alpha\,(1-q)x^2)\,\varphi^2(x), \qquad \varphi(x) = \varphi(-x) \tag{4.10}$$

where $x \in \mathbb{T} := \overline{\{\frac{1}{\sqrt{\alpha\,(q-1)}}\, q^{\mathbb{Z}},\ -\frac{1}{\sqrt{\alpha\,(q-1)}}\, q^{\mathbb{Z}}\ \}}$ *i.e.,* $\beta = \frac{1}{\sqrt{\alpha\,(q-1)}}$*. If the two equations*

$$a_q\varphi = 0, \qquad a_q^+(H_0^q\varphi) = H_1^q\varphi, \tag{4.11}$$

are satisfied then the operators a_q, a_q^+ *are ladder operators to the OPS* $(H_n^q)_{n\in\mathbb{N}_0}$*:*

$$\forall n \in \mathbb{N}_0 : \quad a_q^+(H_n^q\varphi) = H_{n+1}^q\varphi \tag{4.12}$$

$$\forall n \in \mathbb{N}_0 : \quad a_q(H_n^q\varphi) = \frac{\alpha}{q}\,\frac{1-q^n}{1-q}\, H_{n-1}^q\varphi \tag{4.13}$$

$$\forall m, n \in \mathbb{N}_0 : \quad \int_{\mathbb{T}} H_m^q(x)\, H_n^q(x)\, (\varphi(x))^2\, d_qx = \nu_m^2\, \delta_{mn} \tag{4.14}$$

where for $m \in \mathbb{N}$, the numbers ν_m^2 are recursively given by $(\frac{\nu_m}{\nu_{m-1}})^2 = \frac{\alpha}{q}\,\frac{1-q^m}{1-q}$ with $\nu_0^2 = (\varphi, \varphi)$.

Proof. To be ladder operators, it is necessary for a_q, a_q^+ to fulfill the following equations simultaneously:

$$\forall x \in \mathbb{T} : \quad (a_q\varphi)(x) = 0 \ \Leftrightarrow\ (q^{-1}\,(L_qD_q + L_q\xi)\,\varphi)\,(x) = 0, \tag{4.15}$$

$$\forall x \in \mathbb{T} : \quad (a_q^+\varphi)(x) = \alpha x\, \varphi(x) \ \Leftrightarrow\ ((-D_q + \xi R_q)\,\varphi)\,(x) = \alpha x\, \varphi(x). \tag{4.16}$$

(4.15) implies that $(D_q\varphi)(x) + (\xi\varphi)(x) = 0$. Assume now that $\forall x \in \Omega : \ \varphi(x) \neq 0$ where $\Omega \subseteq \mathbb{T}$ is a suitable subset. Then the function ξ is given on Ω by $\xi = -\frac{D_q\varphi}{\varphi}$ and inserting it into (4.16), we obtain

$$\forall x \in \Omega : \ (-\varphi D_q\varphi)(x) - (D_q\varphi)(R_q\varphi)(x) = \alpha x\, \varphi^2(x). \tag{4.17}$$

Using the q-product rule $D_q(fg) = fD_qg + R_qg\, D_qf$ for all $f, g : \mathbb{T} \to \mathbb{C}$, we arrive at the following difference equation on φ:

$$\forall x \in \Omega : \ \varphi^2(qx) = (1 + \alpha\,(1-q)x^2)\,\varphi^2(x), \qquad \varphi(x) = \varphi(-x). \tag{4.18}$$

From (4.18) follows that φ may be in $\mathbb{J} \setminus \{0\}$ if $\mathbb{T}$ has the structure

$$\mathbb{T} := \overline{\{\frac{1}{\sqrt{\alpha\,(q-1)}}\, q^{\mathbb{Z}},\ -\frac{1}{\sqrt{\alpha\,(q-1)}}\, q^{\mathbb{Z}}\}} \tag{4.19}$$

which implies that the (maximal) support of φ^2 resp. φ is:

$$\Omega = \overline{\{\frac{1}{\sqrt{\alpha\,(q-1)}}\, q^{-\mathbb{N}_0},\ -\frac{1}{\sqrt{\alpha\,(q-1)}}\, q^{-\mathbb{N}_0}\}} \tag{4.20}$$

as on $\mathbb{T} \setminus \Omega$, the function φ has to vanish according to the first equation in (4.18). This fixes the parameter β by $\beta = \frac{1}{\sqrt{\alpha\,(q-1)}}$. Let us now choose $\varphi : \mathbb{T} \to \mathbb{R}$ which satisfies (4.18) and $\varphi(x) = 0 \Leftrightarrow x \in \mathbb{T} \setminus \Omega$. Defining

$$\forall x \in \mathbb{T},\ |x| \leq \beta = \frac{1}{\sqrt{\alpha\,(q-1)}} : \quad \xi(x) := (-\frac{D_q\varphi}{\varphi})(x) \tag{4.21}$$

$$\forall x \in \mathbb{T},\ |x| > \beta = \frac{1}{\sqrt{\alpha\,(q-1)}} : \quad \xi(x) := 0, \tag{4.22}$$

we see that the operators

$$a_q^+ = -D_q + \xi\, R_q, \qquad a_q = q^{-1}\,(L_q D_q + L_q\,\xi) \tag{4.23}$$

indeed fulfill equations (4.15) (4.16). We introduce

$$\forall x \in \mathbb{T}: \quad (X_q\psi)(x) := x\psi(x), \tag{4.24}$$

the definition range of X_q being assumed as *maximal* in $\mathbb{J}$. Using

$$\forall f \in B: \quad D_q X_q f = qX_q D_q f + f, \qquad R_q X_q f = qX_q R_q f \tag{4.25}$$

it follows from (4.16) by induction that

$$(a_q^+)^{n+1}\varphi - \alpha q^n\, X_q\,(a_q^+)^n\varphi + \alpha\,\frac{1-q^n}{1-q}\,(a_q^+)^{n-1}\varphi = 0, \qquad n \in \mathbb{N}_0. \tag{4.26}$$

By direct calculation – involving standard methods of analysis – one can verify that all $(a_q^+)^n\varphi \in \mathbb{J}$. Note that the functions $(a_q^+)^n\varphi \in \mathbb{J}$ are well defined in the point $x = 0 \in \mathbb{T}$ as we have $\forall x \in \mathbb{T}:\ ((a_q^+)^n\varphi)(x) = (-1)^n((a_q^+)^n\varphi)(-x),\ n \in \mathbb{N}_0$. Hence (4.12) is valid.

We next address the proof of statement (4.13) using a similar method as in [12, 14]. Let us first calculate the action of the operator a_q on the function $X_q\varphi_n$:

$$qa_qX_q\varphi_n = (L_qD_q + L_q\xi)\,X_q\varphi_n = \tag{4.27}$$

$$L_q(R_q\varphi_n + X_qD_q\varphi_n) + L_q\xi X_q\varphi_n = \varphi_n + L_qX_qD_q\varphi_n + L_q\xi X_q\varphi_n \quad \Leftrightarrow$$

$$a_qX_q\varphi_n = q^{-1}\varphi_n + q^{-2}X_qL_qD_q\varphi_n + q^{-2}X_qL_q\xi\varphi_n = q^{-1}\varphi_n + q^{-1}X_qa_q\varphi_n. \tag{4.28}$$

Stating these identities, we have used the function ξ and the product rule (4.25) for the q-difference operator D_q. Moreover, we have inserted the commutation relation $\forall f \in U:\ L_qX_qf = q^{-1}X_qL_qf$. Let us now consider

$$\varphi_{n+1} - \alpha q^nX_q\varphi_n + \alpha\,\frac{1-q^n}{1-q}\,\varphi_{n-1} = 0, \qquad n \in \mathbb{N}. \tag{4.29}$$

Note that – like in the case of a_q^+ – the operator a_q can be applied to (4.29) yielding

$$a_q\varphi_{n+1} - \alpha q^na_qX_q\varphi_n + \alpha\,\frac{1-q^n}{1-q}\,a_q\varphi_{n-1} = 0, \qquad n \in \mathbb{N}. \tag{4.30}$$

Inserting now (4.28) leads to

$$a_q\varphi_{n+1} - \alpha q^n\,(q^{-1}\varphi_n + q^{-1}X_qa_q\varphi_n) + \alpha\,\frac{1-q^n}{1-q}\,a_q\varphi_{n-1} = 0, \qquad n \in \mathbb{N}. \tag{4.31}$$

We have $a_q\varphi_0 = 0$ and $a_q(\alpha X_q\varphi_0) = \alpha\ q^{-1}\varphi_0 + \alpha\ q^{-1}X_q a_q\varphi_0 = \alpha\ q^{-1}\varphi_0$. Assuming $a_q\varphi_n = \frac{\alpha}{q}\ \frac{1-q^n}{1-q}\ \varphi_{n-1}$ and $a_q\varphi_{n-1} = \frac{\alpha}{q}\ \frac{1-q^{n-1}}{1-q}\varphi_{n-2}$ for $n \in \mathbb{N}$, we obtain for $n \in \mathbb{N}$:

$$a_q\varphi_{n+1} - \alpha q^{n-1}\frac{\alpha}{q}\ \frac{1-q^n}{1-q}\ X_q\ \varphi_{n-1} + \alpha\ \frac{1-q^n}{1-q}\ \frac{\alpha}{q}\ \frac{1-q^{n-1}}{1-q}\varphi_{n-2} - \alpha q^{n-1}\ \varphi_n = 0. \tag{4.32}$$

Rewriting $X_q\varphi_{n-1}$ in terms of (4.29) now leads to $a_q\varphi_{n+1} = \frac{\alpha}{q}\ \frac{1-q^{n+1}}{1-q}\ \varphi_n$ for $n \in \mathbb{N}$ which yields precisely statement (4.13) of Theorem 4.2 via induction.

Let us next verify the orthogonality statement (4.14). For $n \in \mathbb{N}_0$ let us abbreviate

$$\varphi_n := (a_q^+)^n\varphi, \qquad \nu_n := \sqrt{(\varphi_n, \varphi_n)} \neq 0. \tag{4.33}$$

We claim that $(\varphi_m, \varphi_n) = 0$ for $m, n \in \mathbb{N}_0, m \neq n$ which shall be proven by induction. Assume that the elements of the set $\{\varphi_0, \ldots, \varphi_m\}$ are pairwise orthogonal for a fixed $m \in \mathbb{N}_0$. This assumption is trivial for $m = 0$. We now divide the proof of induction into three steps:

1. Let $0 \leq k < m - 1$ where $k \in \mathbb{N}_0, m \in \mathbb{N}$. Then the following equalities hold:

$$(X_q\varphi_m, \varphi_k) = (c_m\varphi_{m+1} + d_m\varphi_{m-1}, \varphi_k) = c_m\ (\varphi_{m+1}, \varphi_k) \tag{4.34}$$

$$(\varphi_m, X_q\varphi_k) = (\varphi_m, c_k\varphi_{k+1} + d_k\varphi_{k-1}) = 0 \tag{4.35}$$

where the sequences $(c_n)_{n\in\mathbb{N}_0}, (d_n)_{n\in\mathbb{N}_0}$ are fixed by $c_n := \alpha^{-1}q^{-n}$ and $d_n := q^{-n}\frac{1-q^n}{1-q}$ for $n \in \mathbb{N}_0$. Due to the fact that X_q is a symmetric operator, we conclude from $(X_q\varphi_m, \varphi_k) = (\varphi_m, X_q\varphi_k)$ that $(\varphi_{m+1}, \varphi_k) = 0$ according to (4.34) and (4.35).

2. Let now $k = m$. Because of φ being an even function, the functions φ_n are even for all even $n \in \mathbb{N}_0$ and the functions φ_m are odd for all odd $m \in \mathbb{N}$. Due to this fact, $(\varphi_m, \varphi_n) = 0$ for all even n and odd m. That's why $(\varphi_m, \varphi_{m+1}) = 0$ for all $m \in \mathbb{N}_0$.

3. Let $k = m - 1$. We first show that $(\varphi_0, \varphi_2) = 0$. Using (4.10), it follows

$$\int_{\mathbb{T}} \varphi^2(qx)\ d_qx = \int_{\mathbb{T}} (1 + (1-q)\alpha x^2)\ \varphi^2(x)\ d_qx = \nu_0^2 + \frac{1-q}{\alpha}\ \nu_1^2 \ \Leftrightarrow \tag{4.36}$$

$$q^{-1}\int_{\mathbb{T}} \varphi^2(x)\ d_qx = \nu_0^2 + \frac{1-q}{\alpha}\ \nu_1^2 \ \Leftrightarrow\ q^{-1}\nu_0^2 = \nu_0^2 + \frac{1-q}{\alpha}\ \nu_1^2 \ \Leftrightarrow\ \nu_1^2 = \frac{\alpha}{q}\ \nu_0^2. \tag{4.37}$$

Expressing φ_2 in terms of φ_0, φ_1 and using (4.12) we obtain

$$\int_{\mathbb{T}} \varphi_0(x)\varphi_2(x)\ d_qx = \int_{\mathbb{T}} (\alpha^2qx^2 - \alpha)\ \varphi^2(x)\ d_qx = q\nu_1^2 - \alpha\nu_0^2 = 0 \tag{4.38}$$

according to (4.37). Hence in total, the functions $\varphi_0, \varphi_1, \varphi_2$ are pairwise orthogonal. Let now $m \in \mathbb{N}$ and $m \geq 2$. We use a similar argumentation as in [5] and obtain:

$$(\varphi_{m+1}, \varphi_{m-1}) = (a_q^+\varphi_m, \varphi_{m-1}) = (\varphi_m, a_q\varphi_{m-1}), \tag{4.39}$$

where the adjointness property is a consequence of the way we have constructed a_q^+, a_q on their chosen domains and of the adjointness relation studied below after (4.43). Due to (4.13), the last equation can be rewritten as

$$(\varphi_{m+1}, \varphi_{m-1}) = (\varphi_m, a_q \varphi_{m-1}) = \frac{\alpha}{q}\,\frac{1-q^{m-1}}{1-q}(\varphi_m, \varphi_{m-2}). \tag{4.40}$$

Due to the result in the first item of the orthogonality proof it now follows that $(\varphi_{m+1}, \varphi_{m-1}) = 0$ as $(\varphi_m, \varphi_{m-2}) = 0$. Taking the steps of this third part together, we have shown that $(\varphi_{m+1}, \varphi_{m-1}) = 0$ if all functions in $\{\varphi_0, \ldots, \varphi_m\}$ are pairwise orthogonal where $m \in \mathbb{N}$. All three parts together finally complete the proof of induction for the orthogonality statement (4.14). Consider now the pairwise orthogonal functions $(\varphi_n)_{n\in\mathbb{N}_0}$ and their normalized versions $(\psi_n)_{n\in\mathbb{N}_0}$. For $m \in \mathbb{N}$ we have $(\psi_{-1} := 0)$:

$$X_q\, \psi_{m-1} = c_{m-1}\,\frac{\nu_m}{\nu_{m-1}}\,\psi_m + d_{m-1}\,\frac{\nu_{m-2}}{\nu_{m-1}}\,\psi_{m-2} \tag{4.41}$$

according to (4.26) where $c_{m-1} := \alpha^{-1} q^{-m+1}$ and $d_{m-1} := q^{-m+1}\,\frac{1-q^{m-1}}{1-q}$. We receive

$$c_{m-1}\,\frac{\nu_m}{\nu_{m-1}} = d_m\,\frac{\nu_{m-1}}{\nu_m} \quad\Leftrightarrow\quad (\frac{\nu_m}{\nu_{m-1}})^2 = \frac{d_m}{c_{m-1}} = \frac{\alpha}{q}\,\frac{1-q^m}{1-q} \tag{4.42}$$

as X_q is a symmetric operator on its maximal domain, using a similar argumentation than after (3.15). This confirms the last statement of Theorem 4.2. Due to Definition 2.3, the operators a_q, a_q^+ have to fulfill the requirement

$$\forall f, g \in B: \quad (a_q^+ f, g) = (f, a_q g). \tag{4.43}$$

Let us check this relation for all finite *real* linear combinations of elements in B by straightforward calculation. We finally obtain:

$$(a_q^+ f, g) = \sum_{n\in\mathbb{Z}} \beta\, q^n (q-1)\, [-\tfrac{f(\beta q^{n+1}) - f(\beta q^n)}{\beta q^{n+1} - \beta q^n} + \xi(\beta q^n)\, f(\beta q^{n+1})]\, g(\beta q^n) +$$

$$\sum_{n\in\mathbb{Z}} \beta\, q^n (q-1)\, [\tfrac{f(-\beta q^{n+1}) - f(-\beta q^n)}{\beta q^{n+1} - \beta q^n} + \xi(-\beta q^n)\, f(-\beta q^{n+1})]\, g(-\beta q^n) =$$

$$\sum_{n\in\mathbb{Z}} \beta\, q^n (q-1)\, [f(\beta q^n)\, \tfrac{g(\beta q^n) - g(\beta q^{n-1})}{\beta q^{n+1} - \beta q^n} + \tfrac{1}{q}\, f(\beta q^n)\, \xi(\beta q^{n-1})\, g(\beta q^{n-1})] +$$

$$\sum_{n\in\mathbb{Z}} \beta\, q^n (q-1)\, [f(-\beta q^n)\, \tfrac{g(-\beta q^n) - g(-\beta q^{n-1})}{-\beta q^{n+1} + \beta q^n} + \tfrac{1}{q}\, f(-\beta q^n)\, \xi(-\beta q^{n-1})\, g(-\beta q^{n-1})] =$$

$$(f,\ q^{-1}\,(L_q D_q + L_q \xi)\, g) = (f, a_q g).$$

One thus arrives at the fact that a_q must necessarily be of type $a_q = q^{-1}\,(L_q D_q + L_q \xi)$ once having chosen $a_q^+ = -D_q + \xi\, R_q$. This verifies indeed the structure in (4.9). In total, Theorem 4.2 is now proven. □

5. SPECTRAL THEORETICAL PROPERTIES AND SUMMARY

In this last section, we want to address some of the spectral theoretical properties of the operators that we have considered throughout the preceeding sections. To do so, let us first review the following definitions for the convenience of the reader:

Definition 5.1. Let Q be a complex Hilbert space and let $T : D(T) \subseteq Q \to Q$ be a linear operator on the subspace $D(T)$ of Q. The **Point Spectrum** $\sigma_p(T)$ of T is the set of all complex values λ to which exists an element $u \in D(T)$ such that $Tu = \lambda u$.

Definition 5.2. Let Q be a complex Hilbert space and $W \subseteq Q$ a dense linear subspace of Q. Let moreover $T : D(T) = W \to Q$ be a linear operator, defined on W. We refer to W by the name **Definition Range** of T. The map $T^* : \; D(T^*) \to Q$ is called the **Adjoint** of T where the linear subspace $D(T^*)$ of Q is defined as follows:

$$y \in D(T^*) \;:\Leftrightarrow\; \exists z \in Q \;\; \forall x \in D(T) : \; (Tx, y) = (x, z), \tag{5.1}$$

$(\circ, \circ)$ denoting the used scalar product of the Hilbert space under consideration. The action of T^* on $y \in D(T^*)$ is defined by

$$T^* y := z. \tag{5.2}$$

We now arrive at the following spectral theorem which summarizes some of the similarities and some of the differences between the continuum case (Section 3) and the discrete case (Section 4) with respect to the properties of the involved linear operators.

Theorem 5.3. *Let A, A^+ and D, X be like in* (3.3) *resp.* (3.4). *We then have*

$$\sigma_p(A^+A) = \mathbb{N}_0, \qquad \sigma_p(X) = \{\ \}. \tag{5.3}$$

Let a_q, a_q^+, X_q be like in (4.9) *and* (4.24) *with $q > 1$. The eigenfunctions of $a_q^+ a_q$ constitute an orthogonal basis in $L^2(\Omega) := \{f \in \mathbb{J} \mid \forall x \in \mathbb{T} \setminus \Omega : f(x) = 0\}$ and we have*

$$\sigma_p(a_q^+ a_q) = \left\{ \frac{\alpha}{q}\, \frac{1 - q^n}{1 - q} \mid n \in \mathbb{N}_0 \right\}. \tag{5.4}$$

Proof. From (3.10) we immediately conclude that $\mathbb{N}_0 \subseteq \sigma_p(A^+A)$. We want to verify that $\sigma_p(A^+A) = \mathbb{N}_0$. Let us define $J_n := (n-1, n) \subset \mathbb{R}$ for $n \in \mathbb{Z}$ and assume that there exists $n \in \mathbb{N}$ such that $\sigma_p(A^+A) \cap J_n \neq \{\ \}$. Let us first show the implication

$$\forall k \in \{1, \ldots, n\} : \quad \sigma_p(A^+A) \cap J_{n+1-k} \neq \{\ \} \;\; \Rightarrow \;\; \sigma_p(A^+A) \cap J_{n-k} \neq \{\ \}. \tag{5.5}$$

To do so, let $\psi \in \mathbb{G}$ such that $A^+A\psi = \lambda\psi$ with $\lambda \in J_n$. This implies $(A^+A\psi, \psi) = (A\psi, A\psi)$ as $\psi \in \mathbb{G}$. On the other hand we recognize $(A^+A\psi, \psi) = (A\psi, A\psi) = \lambda$, hence $A\psi \in \mathbb{G}$ and $A\psi \neq 0$ because otherwise we would have $\lambda = 0$. We then obtain

$$AA^+\, A\psi = \lambda\, A\psi \;\Leftrightarrow\; (A^+A\, A\psi + A\psi) = \lambda\, A\psi \;\Leftrightarrow\; A^+A\, A\psi = (\lambda - 1)\, A\psi \tag{5.6}$$

which means $\sigma_p(A^+A)\cap J_{n-1}\neq\{\ \}$. By successive application of this step we arrive at the statement (5.5) which finally implies that $\sigma_p(A^+A)\cap J_0\neq\{\ \}$. But this however is a contradiction as $(A^+Av,v)=(Av,Av)\geq 0$ for all $v\in\mathbb{G}$. As a consequence we thus obtain $\sigma_p(A^+A)=\mathbb{N}_0$.

We now address the point spectrum of the operator $X:U=\mathbb{G}\to\mathbb{G}$. Assume that $\lambda\in\sigma_p(X)$. Then there exists $f\in U=\mathbb{G}$ such that

$$\forall x\in\mathbb{R}:\quad xf(x)=\lambda\, f(x)\quad\Leftrightarrow\quad(x-\lambda)\, f(x)=0. \tag{5.7}$$

This means that $f(x)$ may be chosen as $f(x)=1$ if $x=\lambda$ and $f(x)=0$ if $x\neq\lambda$. As $\{\lambda\}$ is a Lebesgue set of measure zero, one can identify f with $0\in U=\mathbb{G}$ which is a contradiction to the assumption that f is an eigenvector of X. This directly implies $\sigma_p(X)=\{\ \}$ and (5.3) holds in total.

As a consequence of Theorem 4.2, we recognize that $\Lambda:=\{\ \frac{\alpha}{q}\,\frac{1-q^n}{1-q}\mid n\in\mathbb{N}_0\ \}\subseteq\sigma_p(a_q^+a_q)$. Let now for $n\in\mathbb{N}_0$ the functions ψ_n be the normalized eigenfunctions of $a_q^+a_q$ belonging to the set Λ. Let moreover V be the finite complex span of the vectors in $B_+:=\{e^{\mu}_{-m}\mid m\in\mathbb{N}_0,\mu\in\{+1,-1\}\}$ and $\overline{V}$ the closure of V with respect to $\mathbb{J}$. Let Ω be defined like in (4.20). Let the linear operator $Y:V\to V$ be fixed by the requirement $Ye^{\mu}_{-m}:=\mu q^{-m}e^{\mu}_{-m}$ with $m\in\mathbb{N}_0,\ \mu\in\{+1,-1\}$. The action of the adjoint Y^* of Y is fixed by the equation $(Ye^{\mu}_{-m},e^{\nu}_{-n})_{\overline{V}}=(e^{\mu}_{-m},Y^*e^{\nu}_{-n})_{\overline{V}}$, where $m,n\in\mathbb{N}_0,\ \mu,\nu\in\{+1,-1\}$, the scalar product in $\overline{V}$ being inherited from $\mathbb{J}$. Using the operator X_q from (4.24) with maximal domain in $\mathbb{J}$, the eigenvalue problem for Y^* reads

$$Y^*\sum_{\nu\in\{+1,-1\}}\sum_{n=0}^{\infty}c^{\nu}_n e^{\nu}_{-n}=X_q\sum_{\nu\in\{+1,-1\}}\sum_{n=0}^{\infty}c^{\nu}_n e^{\nu}_{-n}=\lambda\sum_{\nu\in\{+1,-1\}}\sum_{n=0}^{\infty}c^{\nu}_n e^{\nu}_{-n} \tag{5.8}$$

implying that $\sigma_p(Y^*)=\Omega\setminus\{0\}$. Let now the linear operator $T:E\to E$ be fixed by the convention $T\psi_n:=X_q\psi_n$ $(n\in\mathbb{N}_0)$ where E is the finite complex span of all the functions ψ_n. Let T^* be the adjoint of T with respect to the space $\overline{E}$, the scalar product in $\overline{E}$ being inherited from $\mathbb{J}$. From the matrix representation $(T^*\psi_m,\psi_n)_{m,n\in\mathbb{N}_0}$ (all entries are square-summable) it can be concluded that $K:=T^*$ is a symmetric compact operator acting on $\overline{E}$. As K doesn't allow 0 as an eigenvalue, its eigenvectors $(u_n)_{n\in\mathbb{N}_0}$ constitute an orthonormal basis in $\overline{E}$ according to general spectral properties of symmetric compact operators. We have $D(K)=\overline{E}\subseteq L^2(\Omega)=D(Y^*)$, hence $f\in D(K)\Rightarrow f\in D(Y^*)$. We use this to state that on all vectors ψ_n with $n\in\mathbb{N}_0$, the following identity holds:

$$K\ \psi_n=Y^*\ \psi_n. \tag{5.9}$$

As K and Y^* are continuous operators, the last equality means – considering the finite approximations of the eigenvectors u_n by suitable linear combinations of the functions ψ_n – that

$$Ku_n = Y^* u_n = \lambda_n u_n, \qquad n \in \mathbb{N}_0. \tag{5.10}$$

This implies that $B_+ \cap \{ u_n \mid n \in \mathbb{N}_0 \} \neq \{ \}$ as $Y^* e^\mu_{-m} = \mu q^{-m} e^\mu_{-m}$ for $m \in \mathbb{N}_0$ and $\mu \in \{+1, -1\}$. Let us assume now that there exists an element f in B_+ which is not in the set $\{ u_n \mid n \in \mathbb{N}_0 \}$. This leads to the consequence $\forall n \in \mathbb{N}_0 : (f, u_n) = 0$. Due to the basis property of all functions u_n in $\overline{E}$ this implies that for instance $(f, \psi_0) = 0$ which would yield that ψ_0 vanishes identically around zero according to the difference equation for φ in (4.10). This is however not true and therefore $B_+ = \{ u_n \mid n \in \mathbb{N}_0 \}$. As a consequence, all the functions ψ_n constitute an orthonormal basis to $L^2(\Omega)$ which means on the spectral level that $\sigma_p(a_q^+ a_q) = \{ \frac{\alpha}{q} \frac{1-q^n}{1-q} \mid n \in \mathbb{N}_0 \}$. This concludes the proof of Theorem 5.3. □

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