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HYBRID FUZZY SYSTEMS ON TIME SCALES

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ABSTRACT. Hybrid fuzzy dynamic systems on time scales are developed and practical stability of such systems is discussed by the application of Lyapunov-like functions and comparison principle.

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1. INTRODUCTION

It is more realistic to model a phenomena by a dynamic system that incorporates both continuous and discrete times, namely, as an arbitrary closed set of reals known as time scales or measure chains. We refer to [5, 8] for recent results of dynamic systems on time scales. Hybrid systems incorporate both continuous components, usually called plants, which are governed by differential equations, and also digital components such as digital computers, sensors and actuators controlled by programs [3, 7, 8, 11, 12, 13].

The differential equations containing fuzzy-valued functions and interactions with discrete time controllers can be named as hybrid fuzzy differential equations. In this paper we discuss hybrid fuzzy systems on time scales and develop comparison principle and practical stability results. The notion of practical stability is more useful when the desired system may be mathematically unstable and yet the system may oscillate sufficiently near the state that its performance is acceptable [6]. For a basic discussion on fuzzy-valued functions and fuzzy differential equations, we refer to [9, 10, 11]. For details on time scales, we refer to [1, 2].

In this section, we list some basic definitions of fuzzy-valued functions needed for our discussion.

Let $P_k(\mathbb{R}^n)$ denote the family of all nonempty compact, convex subsets of $\mathbb{R}^n$. If $\alpha, \beta \in \mathbb{R}$ and $A, B \in P_k(\mathbb{R}^n)$, then

$$\alpha(A+B) = \alpha A + \alpha B, \quad \alpha(\beta A) = (\alpha\beta)A, \quad 1A = A$$

and if $\alpha, \beta \geq 0$, then $(\alpha+\beta)A = \alpha A + \beta A$. Let $I = [t_0, t_0 + \alpha]$, $t_0 \geq 0$ and $a > 0$ and denote by $E^n = [u : \mathbb{R}^n \to [0, 1]$ such that u satisfies (i) to (iv) mentioned below]:

(i) u is normal, that is, there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$;

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(ii) u is fuzzy convex, that is, for $x, y \in \mathbb{R}^n$ and $0 \le \lambda \le 1$,

$$u(\lambda x + (1-\lambda)y) \ge \min[u(x), u(y)];$$

(iii) u is upper semicontinuous;

(iv) $[u]^0 =$ the closure of $[x \in \mathbb{R}^n : u(x) \ge 0]$ is compact.

For $0 < \alpha < 1$, we denote $[u]^\alpha = [x \in \mathbb{R}^n : u(x) \ge \alpha]$. Then from (i) to (iv), it follows that α-level sets $[u]^\alpha \in P_k(\mathbb{R}^n)$ for $0 \le \alpha \le 1$.

For later purposes, we define $\hat{o} \in E^n$ as $\hat{o}(x) = 1$ if $x = 0$ and $\hat{o}(x) = 0$ if $x \ne 0$.

Let $d_H(A, B)$ be the Hausdorff distance between the sets $A, B \in P_k(\mathbb{R}^n)$. Then we define

$$d[u, v] = \sup_{0 \le \alpha \le 1} d_H[[u]^\alpha, [v]^\alpha],$$

which defines a metric in E^n and (E^n, d) is a complete metric space. We list the following properties of $d[u, v]$ (see [4])

$$d[u + w, v + w] = d[u, v] \quad \text{and} \quad d[u, v] = d[v, u],$$

$$d[\lambda u, \lambda v] = |\lambda| d[u, v],$$

$$d[u, v] \le d[u, w] + d[w, v],$$

for all $u, v, w \in E^n$ and $\lambda \in \mathbb{R}$.

For $x, y \in E^n$ if there exists a $z \in E^n$ such that $x = y + z$, then z is called the H-difference of x and y and is denoted by $x - y$. A mapping $F : I \to E^n$ is differentiable at $t = I$ if there exists a $F'(t) \in E^n$ such that the limits

$$\lim_{h \to 0^+} \frac{F(t+h) - F(t)}{h} \quad \text{and} \quad \lim_{h \to 0^+} \frac{F(t) - F(t-h)}{h}$$

exists and are equal to $F'(t)$. Here the limits are taken in the metric space (E^n, d).

In Section 2, we present some preliminaries on time scales. We discuss comparison theorems and practical stability results of hybrid systems in Sections 3 and 4, respectively.

2. PRELIMINARIES

Let $\mathbb{T}$ be the time scale (any subset of $\mathbb{R}$ with order and topological structure defined in a canonical way) with $t_0 \ge 0$ as the minimal element and no maximal element. Since a time scale $\mathbb{T}$ may or may not be connected, we need the concept of jump operators.

Definition 2.1. The mappings $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined as

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}$$

are called jump operators.

Definition 2.2. A nonmaximal element $t \in \mathbb{T}$ is said to be *right-scattered* (rs) if $\sigma(t) > t$ and *right-dense* (rd) if $\sigma(t) = t$. A nonminimal element $t \in \mathbb{T}$ is called *left-scattered* (ls) if $\rho(t) < t$ and *left-dense* (ld) if $\rho(t) = t$.

Definition 2.3. The mapping $\mu^* : \mathbb{T} \to \mathbb{R}_0^+$ defined by $\mu^*(t) = \sigma(t) - t$ is called *graininess*. When $\mathbb{T} = \mathbb{Z}$, $\mu^*(t) \equiv 1$ and when $\mathbb{T} = \mathbb{R}$, $\mu^*(t) \equiv 0$.

Definition 2.4. The mapping $g : \mathbb{T} \to X$, where X is a Banach space, is called rd-continuous if

(i) it is continuous at each right-dense $t \in \mathbb{T}$,
(ii) at each left-dense point, the left-sided limit $g(t^-)$ exists.

$C_{rd}[\mathbb{T}, X]$ will denote the set of rd-continuous mappings from $\mathbb{T}$ to X. It is clear that a continuous mapping is rd-continuous. However, if $\mathbb{T}$ contains left-dense and right-scattered (ldrs) points, the rd-continuity does not imply continuity, but on a discrete time scale the two notions coincide.

Definition 2.5. A mapping $u : \mathbb{T} \to X$ is said to be differentiable at $t \in \mathbb{T}$, if there exists an $a \in X$ such that for any $\epsilon > 0$, there exists a neighborhood N of t satisfying

$$|u(\sigma(t)) - u(s) - (\sigma(t) - s)a| \leq \epsilon|\sigma(t) - s|$$

for all $s \in N$.

The derivative of u is denoted by u^Δ. Note that if $\mathbb{T} = \mathbb{R}$, $a = du(t)/dt$, and if $\mathbb{T} = \mathbb{Z}$, $a = u(t+1) - u(t)$. In addition, the derivative has the following basic properties:

(i) If u is differentiable at t, then it is continuous at t.
(ii) If u is continuous at t and t is rs, then u is differentiable and

$$u^\Delta(t) = [u(\sigma(t)) - u(t)]/\mu^*(t).$$

Definition 2.6. For each $t \in \mathbb{T}$, let N be a neighborhood of t. Then, we define the generalized derivative (or Dini derivative), $D^+u^\Delta(t)$, to mean that, given $\epsilon > 0$, there exists a right neighborhood $N_\epsilon \subset N$ of t such tat

$$\frac{u(\sigma(t)) - u(s)}{\mu(t,s)} < D^+u^\Delta(t) + \epsilon \quad \text{for} \quad s \in N_\epsilon, \;\; s > t,$$

where $\mu(t, s) \equiv \sigma(t) - s$.

In case t is rs and u is continuous at t, we have, as in the case of the derivative

$$D^+u^\Delta(t) = \frac{u(\sigma(t)) - u(t)}{\mu^*(t)}.$$

Definition 2.7. Let g be a mapping from $\mathbb{T}$ to X. If the mapping $f : \mathbb{T} \to X$ is differentiable on $\mathbb{T}$ and satisfies $f^{\Delta}(t) = g(t)$ for $t \in \mathbb{T}$, then it is called an antiderivative on g on $\mathbb{T}$.

The antiderivative has the following properties:

(i) If $g : \mathbb{T} \to X$ is rd-continuous, then g has an antiderivative $f : t \to \int_r^t g(s)ds$, $r, t \in \mathbb{T}$.

(ii) If the sequence $\{g_n\}_{n\in\mathbb{N}}$ of rd-continuous functions $g_n : \mathbb{T} \to X$ converges uniformly on $[r, s]$ to the rd-continuous function g, then

$$\left(\int_r^s g_n(t)dt\right)_{n\in\mathbb{N}} \to \int_r^s g(t)dt \quad \text{in} \quad X.$$

Definition 2.8. The mapping $f : \mathbb{T} \times X \to X$ is rd-continuous if

(i) it is continuous at each (t, x) with right-dense t, and

(ii) the limits $f(t^-, x) = \lim_{(s,y)\to(t^-,x)} f(s, y)$ and $f_{y\to x}(t, x)$ exist at each (t, x) with left-dense t.

Based on the Definition 2.6, we define for $V \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+]$, $D^+V^{\Delta}(t, x(t))$ to mean that, given $\epsilon > 0$, there exists a right neighborhood N_ϵ of t such that

$$\frac{1}{\mu(t, s)} [V(\sigma(t), x(\sigma(t))) - V(s, x(\sigma(t)) - \mu(t, s)f(t, x(t)))] < D^+V^{\Delta}(t, x(t)) + \epsilon$$

for each $s \in N_\epsilon$, $s > t$. As before, if t is rs and $V(t, x(t))$ is continuous at t, this reduces to

$$D^+V^{\Delta}(t, x(t)) = \frac{V(\sigma(t), x(\sigma(t))) - V(t, x(t))}{\mu^*(t)}.$$

For more details, see [5].

Consider the fuzzy differential equation

$$u' = f(t, u), \quad u(t_0) = u_0 \tag{2.1}$$

when $f \in C[\mathbb{R}_+ \times S(\rho), \mathbb{E}^n]$ and $S(\rho) = [u \in \mathbb{E}^n : d[u, \hat{o}] < \rho]$. Assume that $f(t, \hat{o}) = \hat{o}$, so that we have the trivial solution for (2.1).

Next we state a comparison theorem for fuzzy differential equation (2.1) (see Lakshmikantham and Mohapatra [11, Theorem 4.2.1]).

Theorem 2.9. *Assume that*

(i) $V \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$,

$$|V(t, u_1) - V(t, u_2)| < Ld[u_1, u_2], \quad L > 0$$

and

(ii) $D^+V(t, u) = \lim_{h\to 0} \sup \frac{1}{h}[V(t + h, u + f(t, u)) - V(t, u)] \leq g(t, V(t, u))$, *where* $g \in C[\mathbb{R}^2_+, \mathbb{R}]$.

Then if $u(t)$ is any solution of (2.1) existing on $[t_0, \infty)$ such that $V(t_0, u_0) \leq w_0$, we have

$$V(t, u(t)) \leq r(t, t_0, w_0), \quad t \geq t_0$$

where $r(t, t_0, w_0)$ is the maximal solution of the scalar differential equation

$$w' = g(t, w), \quad w(t_0) = w_0 \geq 0 \tag{2.2}$$

existing on $[t_0, \infty)$.

Similar to Theorem 2.9, we develop a comparison theorem for time scale systems. Consider the fuzzy differential equation on a time scale $\mathbb{T}$,

$$u^\Delta = f(t, u), \quad u(t_0) = u_0, \quad t \in \mathbb{T} \tag{2.3}$$

where $f \in C_{rd}[\mathbb{T} \times S(\rho), E^n]$, and u^Δ denotes the derivative of u with respect to $t \in \mathbb{T}$. We shall assume for convenience, that the solutions $u(t) = u(t, t_0, u_0)$ of (2.3) exist and unique for $t \geq t_0$.

Now we prove the following comparison theorem in terms of Lyapunov function $V(t, u)$. We refer to [5, Theorem 3.1.1] without fuzziness.

Theorem 2.10. *Assume that*

(i) $V \in C_{rd}[\mathbb{T} \times S(\rho), \mathbb{R}_+]$, *$V(t, u)$ satisfies*

$$|V(t, u_1) - V(t, u_2)| < Ld[u, v], \quad L > 0,$$

for $u, v \in S(\rho)$, for each right-dense $t \in \mathbb{T}$,

(ii) *Let*

$$D^+ V^\Delta(t, u(t)) \leq g(t, V(t, u(t))) \tag{2.4}$$

where $g \in C_{rd}[\mathbb{T} \times \mathbb{R}_+, \mathbb{R}]$, $g(t, u)\mu^*(t) + u$ is nondecreasing in u for each (t, u) and the maximal solution $r(t) = r(t, t_0, w_0)$ of the scalar dynamical system

$$w^\Delta = g(t, w), \quad w(t_0) = w_0 \geq 0 \tag{2.5}$$

existing on $\mathbb{T}$, $t \geq t_0$.

Then any solution $u(t) = u(t, t_0, u_0)$ of (2.3) such that $V(t_0, u_0) \leq w_0$ satisfies the estimate

$$V(t, u(t)) \leq r(t, t_0, w_0), \quad t \geq t_0, \quad t \in \mathbb{T}. \tag{2.6}$$

Proof. Let $u(t)$ be any solution of (2.3) existing on $t \in \mathbb{T}$ and $t \geq t_0$. Define $m(t) = V(t, u(t))$ so that $m_0 = V(t_0, u_0) \leq w_0$. Then we apply the induction principle to the statement

$$A(t) : V(t, u(t)) \leq r(t), \quad t \in \mathbb{T}, \quad t \geq t_0.$$

(i) $A(t_0)$ since $V(t_0, u_0) \leq w_0$.

(ii) Let t be rs and $A(t)$ be true. We need to show that $A(\sigma(t))$ is true. By definition, if we set $m(t) = V(t, u(t))$, we see that

$$m(\sigma(t) - r(\sigma(t)) = (D^+ m^\Delta(t) - r^\Delta(t))\mu^*(t) + m(t) - r(t),$$

which, because of $g(t, u)\mu^*(t)$ is nondecreasing in u, and $A(t)$ being true, reduces to

$$\begin{aligned} m(\sigma(t)) - r(\sigma(t)) &\leq [g(t, m(t)) - g(t, r(t))]\mu^*(t) + m(t) - r(t) \\ &\leq 0. \end{aligned}$$

In view of the fact that

$$\frac{m(\sigma(t)) - m(t)}{\mu^*(t)} = \frac{V(\sigma(t), u(\sigma(t))) - V(t, u(t))}{\mu^*(t)}$$

we see that $A(\sigma(t))$ is true. Therefore we have the differential inequality

$$D^+ m^\Delta(t) \leq D^+ V^\Delta(t, u) \leq g(t, V(t, u)), \quad t \geq t_0, \ t \in \mathbb{T}$$

and

$$V(\sigma(t), u(\sigma(t)) \leq r(\sigma(t)).$$

(iii) Let t be rd and N be a right neighborhood of $t \in \mathbb{T}$. We need to show that $A(s)$ is true for $s > t$, $s \in N$. This follows from the comparison Theorem 2.9 for differential equations relative to Lyapunov functions.

(iv) Let t be left-dense and let $A(s)$ be true for all $s < t$. We have to prove that $A(t)$ is true. This follows by the rd continuity of $V(t, u)$, $u(t)$ and $r(t)$.

Therefore by induction principle $A(t)$ is valid for all $t \in \mathbb{T}$. This completes the proof. □

3. COMPARISON THEOREMS

Consider the following hybrid fuzzy dynamic system on a time scale $\mathbb{T}$

$$\begin{cases} u^\Delta = f(t, u, \lambda_k(\tau_k, u_k)), & t \in [\tau_k, \tau_{k+1}], \\ u(\tau_k) = u_k \in S(\rho), & k = 0, 1, 2, \ldots, \end{cases} \tag{3.1}$$

where

(i) $\tau_k \in \mathbb{T}$ for each k, satisfying $0 \leq t_0 \leq \tau_0 < \tau_1, \ldots, < \tau_k \ldots$ and $\tau_k \to \infty$ as $k \to \infty$.

(ii) $f \in C_{rd}[\mathbb{T} \times S(\rho) \times E^n, E^n]$, $\tau_k \in C[\mathbb{T} \times E^n, E^n]$, $S(\rho) = [u \in E^n : d[u, \hat{o}] < \rho]$, and $\hat{o}(t) = 1$ if $t = t_0$ and zero elsewhere.

By a solution of (3.1), we mean the following functions:

$$u(t) = u(t, t_0, u_0) = \begin{cases} u_0(t), & \tau_0 \le t \le \tau_1, \\ u_1(t), & \tau_1 \le t \le \tau_2, \\ \dots & \dots \\ u_k(t), & \tau_k \le t \le \tau_{k+1}, \\ \dots \\ \dots \end{cases}$$

where $u_k(t) = u_k(t, t_k, u_k)$ is a solution of

$$\begin{cases} u_k^{\Delta}(t) = f(t, u_k(t), \lambda_k(\tau_k, u_k)), \\ u_k(\tau_k) = u_k \in S(\rho) \end{cases}$$

$u_k = u_{k-1}(\tau_k, \tau_{k-1}, u_{k-1})$, for each $k = 0, 1, 2, \dots$, and $\tau_k \le t \le \tau_{k+1}$. We assume that the solutions of (3.1) exist and unique for $t \ge t_0$.

We also need the scalar comparison hybrid system

$$\begin{cases} w^{\Delta} = g(t, w, \sigma_k(w_k)), & t \in [\tau_k, \tau_{k+1}], \\ w(\tau_k) = w_k \in \mathbb{R}_+, & k = 0, 1, 2, \dots, \end{cases} \tag{3.2}$$

where $g \in C_{rd}[\mathbb{T} \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}]$ and $\sigma_k \in C[\mathbb{R}_+, \mathbb{R}_+]$. BY the maximal solution $r(t) = r(t, t_0, u_0)$, we mean the following:

$$r(t) = \begin{cases} r_0(t), & \tau_0 \le t \le \tau_1, \\ r_1(t), & \tau_1 \le t \le \tau_2, \\ \dots & \dots \\ r_k(t), & \tau_k \le t \le \tau_{k+1}, \\ \dots & \dots \end{cases}$$

where $r_k(t) = r_k(t, t_k, w_k)$ is the maximal solution of

$$\begin{cases} r_k^{\Delta}(t) = g(t, r_k(t), \sigma_k(w_k)), & \tau_k \le t \le \tau_{k+1}, \\ r_k(\tau_k) = w_k \in \mathbb{R}_+, \\ w_k = r_{k-1}(\tau_k, \tau_{k-1}, w_{k-1}), & \text{for} \quad k = 0, 1, 2, \dots. \end{cases}$$

Next we state the following comparison theorem in terms of Lyapunov-like function V.

Theorem 3.1. *Assume that*

(i) $V \in C_{rd}[\mathbb{T} \times S(\rho), \mathbb{R}_+]$, $V(t, u)$ *satisfies*

$$|V(t, u) - V(t, v)| < Ld[u, v], \quad L > 0,$$

for $u, v \in S(\rho)$, *for each right dense* $t \in \mathbb{T}$, *and for* $\tau_k \le t \le \tau_{k+1}$, $k = 0, 1, 2, \dots$;

(ii) $D^+V^\Delta(t,u) \le g(t, V(t,u(t), \sigma_k(V(t_k,u_k))),\ t \in [\tau_k, \tau_{k+1}]$,

$$g \in C_{rd}[\mathbb{T} \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}], \quad \sigma_k \in C[\mathbb{R}_+, \mathbb{R}_+],$$

$g(t,u,v)\mu^*(t) + u$ is nondecreasing in u for each (t,v), and $\sigma_k(v)$, $g(t,u,v)$ are nondecreasing in v;

(iii) *the maximal solution $r(t) = r(t,t_0,w_0)$ of the scalar hybrid dynamical system*

$$\begin{cases} w^\Delta = g(t, w, \sigma_k(w_k)), & t \in [\tau_k, \tau_{k+1}] \\ w(\tau_k) = w_k \in \mathbb{R}_+, & k = 0,1,2,\dots \end{cases} \tag{3.3}$$

exists for $t \ge t_0$, $t \in \mathbb{T}$.

Then any solution $u(t) = u(t,t_0,u_0)$ of (3.1) such that $V(t_0,u_0) \le w_0$ satisfies the estimate

$$V(t,u(t)) \le r(t,t_0,w_0), \quad t \ge t_0, \quad t \in \mathbb{T}. \tag{3.4}$$

We remark that according to our definition of solutions, (3.4) implies

$$V(t,u(t)) = \begin{cases} V(t,u_0(t)) \le r_0(t), & t_0 \le t \le \tau_1 \\ V(t,u_1(t)) \le r_1(t), & \tau_1 \le t \le \tau_2, \\ \dots \\ V(t,u_k(t)) \le r_k(t), & \tau_k \le t \le \tau_{k+1}, \\ \dots \end{cases}$$

Proof. Let $u(t)$ be any solution of (3.1) existing on $[t_0, \infty)$. Define $m(t) = V(t,u(t))$ so that $m_0 = V(t_0,u_0) \le w_0$. Then by applying the induction principle similar in the proof of Theorem 2.10, we get

$$D^+m^\Delta(t) \le D^+V^\Delta(t,u) \le g(t, V(t,u), \sigma_k(V(\tau_k,u_k)))$$

for $\tau_k \le t \le \tau_{k+1}$ where $m_k = V(t_k, u(t_k))$.

For $t \in [\tau_0, \tau_1]$, since $m(t_0) = V(t_0,u_0) \le w_0$, by Theorem 2.10, we obtain

$$V(t,u_0(t)) \le r_0(t,t_0,w_0), \quad \tau_0 \le t \le \tau_1,$$

where $r_0(t) = r_0(t,t_0,w_0)$ is the maximal solution of

$$\begin{cases} w_0^\Delta = g(t, w_0, \sigma_0(w_0)), & \tau_0 \le t \le \tau_1, \\ w_0(\tau_0) = w_0 > 0, \end{cases}$$

and $u_0(t)$ is the solution of

$$\begin{cases} u_0^\Delta(t) = f(t, u_0(t), \lambda(\tau_0, u_0)) \\ u(t_0) = u_0. \end{cases}$$

Similarly, for $t \in [\tau_1, \tau_2]$, it follows that

$$V(t, u_1(t) \le r_1(t, \tau_1, w_1), \quad \tau_1 \le t \le \tau_2,$$

where $w_1 = r_0(t, t_0, w_0)$, $r_1(t, \tau_1, w_1)$ is the maximal solution of

$$\begin{cases} w_1^{\Delta}(t) = g(t, w_1(t), \sigma_1(w_1)), & \tau_1 \le t \le \tau_2, \\ w_1(\tau_1) = w_1 \ge 0, \end{cases}$$

and $u_1(t)$ is the solution of

$$\begin{cases} u_1^{\Delta}(t) = f(t, u_1(t), \lambda_1(\tau_1, u_1)), & \tau_1 \le t \le \tau_2, \\ u_1(\tau_1) = u_1. \end{cases}$$

Similarly proceeding, we can obtain

$$V(t, u_k(t)) \le r_k(t, \tau_k, w_k),$$

where $u_k(t)$ is the solution of

$$\begin{cases} u_k^{\Delta}(t) = f(t, u_k(t), \lambda_k(\tau_k, u_k)), & \tau_k \le t \le \tau_{k+1}, \\ u_k(t) = u_k, \end{cases}$$

$u_k = u_{k-1}(\tau_k, \tau_{k-1}, u_{k-1})$ and $r_k(t, \tau_k, w_k)$ is the maximal solution of

$$\begin{cases} w_k^{\Delta}(t) = g(t, w_k(t), \sigma_k(w_k)), & \tau_k \le t \le \tau_{k+1}, \\ w_k(\tau_k) = w_k, \end{cases}$$

where $w_k = r_{k-1}(\tau_k, \tau_{k-1}, w_{k-1})$.

Thus defining $r(t, t_0, w_0)$ as the maximal solution of (3.2) as

$$r(t, t_0, w_0) = \begin{cases} r_0(t, t_0, w_0), & \tau_0 \le t \le \tau_1, \\ r_1(t, \tau_1, w_1), & \tau_1 \le t \le \tau_2, \\ \dots & \dots \\ r_k(t, \tau_k, w_k), & \tau_k \le t \le \tau_{k+1}, \\ \dots & \dots \end{cases}$$

and taking $w_0 = V(t, u_0)$, we obtain the estimate

$$V(t, u(t)) \le r(t), \quad t \ge t_0, \quad t \in \mathbb{T}.$$

□

4. PRACTICAL STABILITY

In this section, we discuss practical stability for the solutions of (3.1).

Definition 4.1. The hybrid fuzzy dynamic system is said to be

(i) practically stable, if given (λ, A) with $0 < \lambda < A$, we have $d[u_0, \hat{o}] < \lambda$ implies $d[u, \hat{o}] < A$, $t \ge t_0$, $t \in \mathbb{T}$;
(ii) practically quasi-stable, if given $(\lambda, B, T_0) > 0$, we have $d[u_0, \hat{o}] < \lambda$ implies $d[u(t), \hat{o}] < B$, $t \ge t_0 + T_0$, $t_0 + T_0 \in \mathbb{T}$;

(iii) strongly practically stable, if (i) and (ii) hold simultaneously;

(iv) practically asymptotically stable, if (i) holds and for any $\epsilon > 0$ there exists $T_0 > 0$ such that $t_0 + T_0 \in \mathbb{T}$ and $d[u_0, \hat{o}] < \lambda$ implies, $d[u(t), \hat{o}] < A$, $t \geq t_0 + T_0$.

Similarly we can define corresponding notions for comparison hybrid dynamic system (3.2).

In the following, we provide sufficient conditions for practical stability of hybrid fuzzy dynamic system (3.1).

Theorem 4.2. *Assume that* λ, A *are given such that* $0 < \lambda < A$ *and conditions* (*i*) *and* (*ii*) *of Theorem* 3.1 *hold with* $S(\rho)$ *is replaced by* $S(A)$. *Suppose further that* $V(t, u)$ *satisfies*

$$b(d[u, \hat{o}]) \leq V(t, u) \leq a(d[u, \hat{o}]), \quad (t, u) \in \mathbb{T} \times S(A),$$

where $a, b \in \mathcal{K} = [d \in C[\mathbb{R}_+, \mathbb{R}_+]$ *such that* $d(0) = 0$ *and* $d(v)$ *is increasing in* $v]$ *and* $a(\lambda) < b(A)$. *Then the practical stability properties of hybrid dynamic system* (3.2) *imply the corresponding practical stability properties of the hybrid fuzzy dynamic system* (3.1).

Proof. First we assume that (3.2) is practically stable. Then given $(a(\lambda), b(A))$, it follows that $0 \leq w_0 < a(\lambda)$ implies $w(t, t_0, w_0) < b(A)$, $t \geq t_0$, $t \in \mathbb{T}$, where $w(t, t_0, w_0)$ is any solution of (3.2). Let $d[u_0, \hat{o}] < \lambda$. Then we claim that $d[u(t), \hat{o}] < A$, $t \geq t_0$, $t \in \mathbb{T}$, where $u(t) = u(t, t_0, u_0)$ is the solution of (3.1) with $d[u_0, \hat{o}] < \lambda$. If this is not true, there would exist a $t_1 > t_0$, $t_1 \in \mathbb{T}$ and a solution $u(t) = u(t, t_0, u_0)$ with $d[u_0, \hat{o}] < \lambda$ such that

$$d[u(t_1), \hat{o}] \geq A \quad \text{and} \quad d[u(t), \hat{o}] < A, \quad t_0 \leq t < t_1.$$

This shows that $V(t_1, u(t_1)) \geq b(d[u(t_1), \hat{o}]) \geq b(A)$.

Choose $V(t_0, u_0) = w_0$ and make the special choice of u_k and w_k as in Theorem 3.1, to arrive at

$$V(t, u(t)) \leq r(t, t_0, V(t_0, w_0)), \quad t_0 \leq t \leq t_1.$$

But then we would have

$$\begin{aligned} b(A) &\leq b(d[u(t_1), \hat{o}]) \leq V(t_1, u(t_1)) \leq r(t_1, t_0, V(t_0, u_0)) \\ &\leq r(t_1, t_0, a(d[u_0, \hat{o}]) \leq r(t_1, t_0, a(\lambda)) < b(A), \end{aligned}$$

which is a contradiction. Hence it follows that the hybrid fuzzy dynamic system (3.1) is practically stable.

Next we prove that (3.1) is strongly practically stable for $(\lambda, A, B, T_0) > 0$, $\lambda < A$, $t_0 + T_0 \in \mathbb{T}$. To prove this, we assume that (3.2) is strongly practically stable for $(a(\lambda), b(A), b(B), T_0) > 0$. This means that we need to prove only quasi-stability of (3.1). Since (3.2) is practically quasi-stable, we get $0 \leq w_0 \leq a(\lambda)$ implies

$w(t_1, t_0, w_0) < b(B)$, $t \geq t_0 + T_0$, where $w(t, t_0, w_0)$ is any solution of (3.2). Suppose that $d[u_0, \hat{o}] < \lambda$ so that we have $d[u(t), \hat{o}] < A$, $t \geq t_0$, because of practical stability of (3.2). As a result, it follows that the estimate

$$V(t, u(t)) \leq r(t, t_0, w_0), \quad t \geq t_0$$

is valid and this leads for $t \geq t_0 + T_0$,

$$\begin{aligned} b(d[u(t), \hat{o}]) &\leq V(t, u(t)) \leq r(t, t_0, V(t_0, u_0)) \\ &\leq r(t, t_0, a(\lambda)) < b(B). \end{aligned}$$

Thus we see that $d[u(t), \hat{o}] < B$, $t \geq t_0 + T_0$, whenever $d[u_0, \hat{o}] < \lambda$ and therefore, strong practical stability of (3.1) follows. Similarly practical asymptotic stability of (3.1) can be proved. The proof is thus complete. □

REFERENCES

[1] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston 2001.

[2] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston 2001.

[3] F. H. Clark, Y. Ledyaev, S. Yum, R. I. Stern, and P. R. Wolenski, *Nonsmooth Analysis and Control Theory*, Springer-Verlag, New York 1998.

[4] O. Kalvea, Fuzzy differential equations, *Fuzzy Sets and Systems* 25: 301–317, 1987.

[5] V. Lakshmikantham, S. Sivasundaram, and B. Kaymakçalan, *Dynamic Systems on Measure Chains*, Kluwer Academic Publishers, Dordrecht 1996.

[6] V. Lakshmikantham, S. Leela, and A. A. Martynyuk, *Practical Stability of Nonlinear Systems*, World Scientific, Singapore 1990.

[7] V. Lakshmikantham and X. Z. Liu, Impulsive hybrid systems and stability theory, *Dynam. Systems Appl.* 7:1–9, 1998.

[8] V. Lakshmikantham and A. S. Vatsala, Hybrid systems on time scales, *J. Comput. Appl. Math.* 141:227–335, 2002. Special Issue on "Dynamic Equations on Time Scales".

[9] V. Lakshmikantham and R. N. Mohapatra, Basic properties of solutions of fuzzy differential equations, *Nonlinear Stud.* 8:113–124, 2001.

[10] V. Lakshmikantham and S. Leela, Stability theory of fuzzy differential equations via differential inequalities, *Math. Inequal. Appl.* 2:551–559, 1999.

[11] V. Lakshmikantham and R. N. Mohapatra, *Theory of Fuzzy Differential Equations and Inclusions*, Taylor and Francis, United Kingdom 2003.

[12] A. Nerode and W. Kohn, *Models in Hybrid Systems*, Lecture Notes in Computer Science 36, Springer-Verlag, Berlin 1993.

[13] A. N. Michel, Recent trends in stability analysis of hybrid dynamic systems, *IEEE Trans. Circuits Systems* 45:120–134, 1999.

[14] M. Sambandham, Perturbed Lyapunov-like functions and hybrid fuzzy differential equations, *Int. J. Hybrid Systems* 2: 23–34, 2002.