7. Find examples of $2 \times 2$-matrices with:
   (a) $A^2 = -I$ ($A$ having only real entries);
   (b) $B^2 = 0$ (but $B \neq 0$);
   (c) $CD = DC$ (but $CD \neq 0$);
   (d) $EF = 0$ (neither $E$ nor $F$ having any zero entries);
   (e) $AB = AC$ but $B \neq C$;
   (f) $A + B$ is not invertible but $A$ and $B$ are;
   (g) $A + B$ is invertible but $A$ and $B$ are not;
   (h) $A$ and $B$ are symmetric but $AB$ is not.

8. Let $A$ be any matrix. Show that $AA^T$ and $A^T A$ are both symmetric.

9. A real $2 \times 2$-matrix is called symplectic if $A^TJA = J$, where $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Characterize symplectic matrices in terms of their entries.

10. If $A$, $B$, and $A + B$ are invertible, show that $A^{-1} + B^{-1}$ is invertible and find a formula for its inverse in terms of $A$, $B$, $A + B$ and their inverses.

11. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Find all matrices $M$ for which $AM = A$.

12. Prove that diagonal matrices of the same order commute.

13. Let $D$ be an arbitrary diagonal matrix. When is $D$ invertible? If it is invertible, what is $D^{-1}$?

14. Let $A$ and $D$ be square matrices of the same size. Assume that $D$ is diagonal. Describe how $AD$ looks like. How about $DA$?

15. Let $A$ be a matrix of size $m \times n$. Find a matrix $P$ such that $P$ multiplied with $A$ exchanges the $i$th row and the $j$th row of $A$. What needs to be done if the $i$th column and the $j$th column of $A$ should be exchanged?

16. Suppose that $(I + A)^{-1}A = B$ holds for two matrices $A$ and $B$.
   (a) Prove that $A$ and $B$ commute.
   (b) Prove that, if $B$ is invertible and diagonal, then also $A$ is invertible and diagonal.

17. Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. Find all vectors $v$ that satisfy $Av = 0$.

18. Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 0 \\ 5 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 6 \\ 10 \\ 14 \end{bmatrix}$. Find numbers $a$, $b$, and $c$ with $av_1 + bv_2 + cv_3 = 0$. 