23. Determine whether the following \((V, +, \cdot)\) are real vector spaces and justify your claims.

(a) \(V = \left\{ \begin{bmatrix} 1 \\ x \end{bmatrix} : x \in \mathbb{R} \right\}, \begin{bmatrix} 1 \\ x \end{bmatrix} + \begin{bmatrix} 1 \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ x+y \end{bmatrix}, c \cdot \begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ cx \end{bmatrix} \);

(b) \(V = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2 + x_2 \\ y_1 + 2 + y_2 \end{bmatrix}, c \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix} \);

(c) \(V = (0, \infty), x + y = xy, c \cdot x = x^c;\)

(d) \(V\) the set of all invertible \(2 \times 2\)-matrices, with usual matrix addition and multiplication of a matrix by a scalar;

(e) \(V\) the set of all non-invertible \(2 \times 2\)-matrices, with usual matrix addition and multiplication of a matrix by a scalar.

24. Which of the following sets are subspaces of \(\mathbb{R}^3\)? Again, justify your claims.

(a) The set of vectors in \(\mathbb{R}^3\) with first component 0;

(b) The set of vectors in \(\mathbb{R}^3\) with last component 4;

(c) The set of vectors in \(\mathbb{R}^3\) whose components multiplied together gives zero;

(d) The set of vectors in \(\mathbb{R}^3\) whose first two components are the same;

(e) The set of all linear combinations of the two vectors \(\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}\) and \(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\);

(f) The set of vectors \(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\) whose components satisfy \(4a - b + 2c = 0;\)

(g) The set of vectors \(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\) whose components satisfy \(4a - b + 2c - 4 = 0.\)

25. Let \(V\) be the vector space consisting of all \(3 \times 3\)-matrices (with usual matrix addition and multiplication of a matrix by a scalar). Find the smallest subspace which contains all symmetric matrices and all lower triangular matrices. What is the largest subspace which is contained in both of these subspaces?

26. Let \(V\) be a vector space and let \(U_1\) and \(U_2\) be subspaces. Prove that \(U_1 \cap U_2\) is also a subspace of \(V\). How about \(U_1 \cup U_2\)?